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Essays in Dynamic General Equilibrium Theory

Festschrift for David Cass

With 23 Figures
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Essays in General Equilibrium Theory Festschrift for David Cass

Guest Editors' Introduction

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This book is a supplement to the special edition of *Economic Theory* honoring David Cass on the 30$^{\text{th}}$ anniversary of his joining the faculty at the University of Pennsylvania's Economics Department. As with the *ET* issue, the contributions to this volume are, for the most part, from Dave's students or co-authors, and we hope they communicate both to Dave and to the economics profession generally the high regard those of us who have trained under Dave's tutelage or worked with him on research have for him.

Most scientists would be happy to have had one major, influential idea over the course of their careers. Dave Cass has had three, and is still going strong.

His first major contribution to economics was the characterization of optimal growth trajectories in his thesis work under Hirofumi Uzawa's supervision. The celebrated Cass criterion for optimal time paths in the one good growth model quickly followed. The essence of this work is the search for price characterizations of efficiency for dynamic time paths, an effort that directly pointed the way to the subsequent full dynamic decentralization of the neoclassical optimal growth model, a fact that permits its use for modeling a wide range of business cycle and other macroeconomic phenomena. Accordingly, Dave is rightly honored, in conjunction with Tjalling Koopmans, as one of the fathers of dynamic macroeconomic analysis.

Dave's second contribution – the notion of a so-called sunspot equilibrium in dynamic economies which he developed jointly with Karl Shell – is also the stuff of legend, and grew out of his long and productive collaboration with Karl at Penn. The early impetus for Dave's interest in this topic stemmed from work he did with Manny Yaari on overlapping generations models, and from his early acquaintance with Bob Lucas at Carnegie Mellon and Lucas's seminal work on rational expectations in dynamic economic models. To quote from the interview with Dave by Spear and Wright in *Macroeconomic Dynamics*

I wasn't so interested in macro, but what struck me, and this is related to some of my later work, was the assumption that [Lucas] made to solve for equilibrium, that the state variables were obvious.... Bob and I had some long discussions, and I would say, "Well Bob, why is this the actual state space in this model?" That question came up ... after I came to Penn. At some point Karl [Shell] and I started talking about that and we developed what we called the idea of sunspots. (Spear and Wright [8])

In addition to raising troubling questions about what the right state space was for dynamic stochastic economies, the notion of sunspot equilibrium raised a number of deep questions about the overall determinacy of economic equilibria and the role of the welfare theorems in the occurrence or non-occurrence of sunspot equilibria. These questions spawned a large literature on determinacy in dynamic economies in which the welfare theorems broke down. These include overlapping generations models, growth models with externalities or taxes, and models in which asset markets were incomplete. All were shown to allow the existence of sunspot equilibria. And, in a suitable twist of intellectual fate, macroeconomists have more recently begun to explore the question of whether sunspots can provide a more plausible source of fluctuations in dynamic equilibrium models than the conventional aggregate productivity disturbances.

Dave's third major contribution to economic theory was his work on general equilibrium with incomplete markets, work which grew out of his exploration of the question of existence of sunspot equilibria in models with incomplete asset markets. Dave's follow-on work on existence and determinacy of general equilibrium in models with incomplete asset markets spawned another large literature which has come to be known simply as GEI.

The earliest work on market incompleteness goes back to Arrow in the 1950's, Diamond in the mid-'60's and a number of related papers in the finance literature between the late 1950's and early '70's (Geanakoplos [4] provides an excellent survey of this literature). The canonical GEI model was formulated by Radner in the early 1970's (Radner [7]) in a paper which also pointed up one of the fundamental puzzles about models with incomplete markets: the possible loss of dimensionality in the span of the asset payoffs as prices vary.

This potential for non-existence of equilibrium (which was formally developed in Hart's [6] counterexamples to existence of equilibrium) left the literature in limbo for almost a decade, until Dave's work on existence in economies with purely financial assets pointed the way out. As Geanakoplos notes

> Suddenly in the middle 1980s the pure theory of GEI fell into place. In two provocative and influential papers, Cass [1,2] showed that the existence of equilibrium could be guaranteed if all the assets promise delivery in fiat money, and he gave an example showing that with such financial assets there could be a multiplicity of equilibrium. Almost simultaneously Werner [9] also gave a proof of existence of equilibrium with financial assets, and Geanakoplos and Polemarchakis [5] showed the same for economies with real assets that promise delivery in the same consumption good. (Geanakoplos [4])

This work was followed very quickly by results showing that the nonexistence problem pointed out by Hart was not generic, and led ultimately to the generic existence results of Duffie and Shafer [3], and again spawned a new literature looking positively at the welfare implication of market incompleteness, and normatively at issues of asset engineering.

In the course of making these contributions, Dave has worked with a large group of coauthors, including (to date): Y. Balasko, L. Benveniste, G. Chichilnisky, A. Citanna, R. Green, M. Majumdar, T. Mitra, M. Okuno, A. Pavlova, H. Polemarchakis, K. Shell, P. Siconolfi, S. Spear, J. Stiglitz, A. Villanacci, H.-M. Wu, M. Yaari, and I. Zilcha. Dave's graduate students (to date) include S. Chae, A. Citanna, J. Donaldson, R. Forsythe, F. Kydland, Y. W. Lee, M. Lisboa, A. Pavlova, T. Pietra, P. Siconolfi, S. Spear, S. Suda, J-M. Tallon, and A. Villanacci. Those of us who have worked with Dave and/or under his tutelage as graduate students have benefited tremendously from his razor-sharp analytic mind, from his willingness to work at understanding problems we have posed to him, or new methodological techniques we have discovered, and (perhaps most importantly) from his no-nonsense approach to doing science.

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Monopoly Power and the Firm's Valuation: A Dynamic Analysis of Short versus Long-Term Policies*

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Summary. Recent anti-trust cases exacerbated the concerns of investors regarding the effects of a firm's monopoly power on its production choice, shareholder value, and the overall economy. We address this issue within a dynamic equilibrium model featuring a large monopolistic firm whose actions not only affect the price of its output, but also effectively influence the valuation of its stock. The latter renders time-inconsistency to the firm's dynamic production choice. When the firm is required to pre-commit to its strategy, the ensuing equilibrium is largely in line with the predictions of the textbook monopoly model. When the firm behaves in a time-consistent manner, however, the predictions are strikingly at odds. The trade-off between current profits and the valuation of future profits induces the firm to increase production beyond the competitive benchmark and cut prices. This policy may result in destroying shareholder value, and does indeed fully wipe out the firm's profit in the limit of the decision-making interval shrinking to zero, in line with the Coase conjecture.

1 Introduction

The pervasiveness of monopoly power, across many product categories in almost every part of the world economy, has long pre-occupied economists and lawmakers. There is also undisputed evidence that monopoly power is widespread amongst firms dominant enough to matter at an economy-wide

*We thank Steve Spear and the anonymous referees for helpful suggestions. We are also grateful to Franklin Allen, Dave Cass, Peter DeMarzo, Bernard Dumas, Ron Giammarino, Rich Kihlstrom, Leonid Kogan, Branko Urosevic, Dimitri Vayanos, seminar participants at Boston University, University of Colorado at Boulder, Columbia University, MIT, University of Pennsylvania, Princeton University, American Finance Association Meetings, and European Finance Association Meetings for valuable comments. All errors are solely our responsibility.

level, the so-called "bellwethers." This is evidenced by the series of ongoing anti-trust cases brought about by the U.S. government against, for example, IBM (1969 - 1982), AT&T (1974 - 1982), Microsoft (1994 - 1995, 1998 - present), and Bell Atlantic (1996 - 1997). Apparently, there is concern that these large monopolists may not just exert power in their own product markets, but their actions may impact the overall economy. For example, as argued in the Microsoft case, US consumers and businesses alike feared to become critically dependent on Microsoft products. There are well-known dominant players outside the US, too. For instance, OPEC's production decisions appear to affect not only the oil producing countries' economies, but also the overall world economy. While the surveys of imperfect competition by Hart (1985) and Bonanno (1990) have highlighted the importance of a general equilibrium analysis of such large monopolists, most studies of monopoly behavior have been undertaken at a partial equilibrium level assuming no impact on other markets. A true general equilibrium approach must face up to the challenge of accounting for the so-called "feedback" or "Ford effect," as once argued by Henry Ford: through the influence of its own actions, a non-price-taking firm may affect the wealth of its customers, and hence the demand for the firm's product. Finally, although there is growing work on general equilibrium asset pricing with market imperfections (e.g., see the survey by Sundaresan (2000)), the consideration of monopoly power is still missing in this literature.

Our primary objective in this paper is to investigate the optimal behavior of a monopolist who has sufficient power to impact economy-wide pricing. We model the extreme case of an economy containing a single monopolistic firm who then, at a general equilibrium level, impacts the overall price of consumption in the economy. Beyond the traditional assumption that its actions impact the price of its own good, the firm's actions also influence the valuation of its stock. As will become evident from our analysis, the dynamic setting we employ leads to a distinction between short-term and long-term policies of the monopolistic firm. (For a competitive firm, there is no such distinction.) Part of our emphasis will be to study the differences between the ensuing equilibria under short versus long-term policies.[3]

We adopt a familiar Robinson-Crusoe formulation of a discrete-time production finite-horizon economy populated by a representative consumer-investor-worker and a representative firm. The consumer derives utility from

[3]Firms' short-termism is receiving increasing attention from both academics and practitioners. For example, Business Week (09/13/1999) voices the often-quoted complaint that Wall Street exerts increasing pressure on many firms for short-term results due to a shrinking of investors' time horizons. The article points out that a share in AT&T is held for an average of 1.1 years, down from 3 years in 1990; a share in General Motors is held for an average of 1 year, down from 2 years in 1990. Firms' short-termism is also frequently blamed on ever increasing managerial turnover. Anecdotal evidence and recent academic literature (Allen and Gale (2000), Palley (1997)) seem to agree that revolving management may be damaging for a long-lived firm.

consumption and leisure, simultaneously invests in financial markets (including the firm's stock), and earns a labor income. The consumer's labor is demanded by the firm as the sole input to a stochastic non-constant-returns-to-scale production technology, producing the only good in the economy. We assume the objective of this firm is to maximize its market value, or the present value of its expected profits. The firm has monopoly power in the good market, in that it takes account of the impact (via market clearing) of its production plan on the price of its output. In our setting, the monopoly power manifests itself as an impact of the firm's labor demand on the state prices (or the pricing kernel). This manipulation of state prices results in time-inconsistency of the firm's production strategy, in that it has an incentive to deviate from the initial plan at a later date. To rule out time-inconsistency, we focus on two distinct types of monopolistic strategies: a "pre-commitment" strategy in which the firm initially chooses a plan to maximize its initial value, and thereafter cannot deviate from that plan; and a "time-consistent" strategy in which the firm chooses a plan each period, maximizing the value at that period, taking into account the re-adjustments that it will make in the future. Consistently with the literature (e.g., Blanchard and Fischer (1989, §11.4)), we interpret the time-consistent strategy as a short-sighted or short-term strategy, and the pre-commitment one as long-term.

Solving for the pre-committed monopolist's strategy reveals his optimal plan and the extent of his monopoly power to be driven by the concurrent profit, the marginal product of labor, and the consumer's attitude towards risk over consumption. The most immediate implications on the ensuing dynamic equilibrium are consistent with the predictions of textbook static monopoly models: lower good output and lower labor input, and higher price of consumption than the competitive counterpart. However, in contrast to the textbook case, profits and the firm's value can be either higher or lower. This arises because it is the firm's initial value the monopolist maximizes, not the profits nor the value at later times. By restricting production, he moves away from the competitive, profit maximizing, production plan.

The optimal behavior of the time-consistent monopolist contrasts sharply with that of the pre-committed monopolist. His optimal production plan and his monopoly power are driven by the negative of the ex-dividend stock price, in place of concurrent profits. In attempting to maximize the current stock valuation, this monopolist trades off between today's profit and the ex-dividend stock price; hence the appearance of the ex-dividend price. In direct opposition to the pre-commitment case, the equilibrium good output and labor demand are higher than the competitive case, while the price of consumption is lower. It is in this short-term monopolist's interest to depress the current price of consumption, so as to boost today's stock price. Yet more strikingly, the profits in every period are decreased in the monopolist's presence, and may even go negative, while the firm's value can be either lower or higher than in the competitive benchmark. We also argue that our main conclusions regarding the time-consistent equilibrium hold true in the infinite horizon limit of our economy within a parametric example. To further explore the robustness of these results, we provide an alternative, albeit extreme, form of a production opportunity (constant-returns-to-scale) under which the monopoly power vanishes, and the solution coincides with the competitive.

It is well-recognized in monopolistic models of durable goods, that absent commitment the monopolistic firm may in a sense "compete with itself" across different time periods, and weaken its monopoly power. The Coase (1972) conjecture proposes that as the time interval between successive decisions is reduced to an infinitesimal length, this intertemporal competition will drive the firm's profits to zero. Since our long-lived firm's stock resembles a durable good, it is of interest to explore the limiting case of our economies where the decision-making time interval shrinks to zero. While the competitive and the pre-commitment monopolistic equilibrium retain their basic discrete-time structure and implications, we find the time-consistent equilibrium to tend to the limit of zero profits and hence zero firm's value at all times. Hence, under the time-consistent scenario, monopoly power destroys shareholder value. However, within our framework, this zero profit limit does not coincide with the competitive solution.

The importance of imperfect competition is, of course, well-recognized in many areas of theoretical and applied research. The standard textbook treatment of monopoly (Mas-Colell, Whinston and Green (1995, Chapter 12), Tirole (1988, Chapter 1)) considers a profit-maximizing firm which is the only producer of a good (and has no influence on other markets), in a static partial equilibrium setting. The main implications of the textbook monopolist are restriction of production and a raising of the price of output in the market for its product. While this behavior is consistent with the equilibrium implications of our pre-committed monopolist, it is at odds with those of the time-consistent monopolist. The distinction is due to our accounting for the economy-wide impact of the firm's decision and in particular for the impact on its stock price valuation.

In that regard, the most closely related work in a general equilibrium setting where the valuation of financial securities is affected by market power are the works of Basak (1997) and Kihlstrom (1998). These authors consider a single large consumer-investor acting as a non-price-taker in securities markets, in a Lucas (1978)-type pure-exchange setting.[4] While the monopolists in Kihlstrom and Basak select a consumption-investment plan to maximize utility, our monopolist selects a production plan to maximize the stock price of his firm. In this respect, ours is much closer to the standard textbook monopolist. Basak demonstrates that the non-price-taker acting as a price-leader in all markets manifests itself as a dependence of the state prices on the agent's consumption choice. Kihlstrom relates the dynamic security price choice of the monopolist to the Coase (1972) conjecture, and shows that the inability

[4] See also Lindenberg (1979) and Grinblatt and Ross (1985) for related analysis within a static mean-variance framework.

of the monopolist to commit to the future (second period) price quote reduces his monopoly rents. Consequently, the first period security price is less than in the commitment scenario (but is still higher than the competitive price). In the spirit of Kihlstrom, but with an additional moral hazard problem of the monopolist, is the dynamic model of DeMarzo and Urosevic (2001).[5] Absent commitment, they demonstrate an analog of the Coase conjecture in the continuous-time limit of their economy.

The rest of the paper is organized as follows. Section 2 describes the economy. Section 3 characterizes equilibrium in the economy with a monopolistic firm for the cases when the firm can commit to its future production plan and when it cannot. It also presents comparison of the resulting equilibrium quantities to those of a benchmark competitive economy. In Section 4, we take the economy to its continuous-time limit and explore the Coase conjecture in the context of our economy. Section 5 concludes and the Appendix provides all proofs as well as discussions of alternative choices of the numeraire and the case of a monopolistic-monopsonistic firm.

2 The Economy

We consider a simple Robinson-Crusoe production economy with a representative firm and a representative consumer-investor-worker. We make the standard assumption that the consumer-investor-worker represents a continuum of identical atomistic agents who take prices as given and cannot act strategically. The economy has a finite horizon $[0, T]$, in which trading takes place at discrete times $t = 0, \dots, T$. There is a single consumption good serving as the numeraire (other choices of the numeraire are discussed in Remark 1). Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t;\, t = 0,\, 1, \dots, T\}, \mathcal{P})$ generated by a production shock process ε. All stochastic processes are assumed adapted to $\{\mathcal{F}_t;\, t = 0,\, 1, \dots, T\}$, all stated (in)equalities involving random variables hold $\mathcal{P}$-almost surely. We assume all processes and expectations are well-defined, without explicitly stating the required regularity conditions.

The financial investment opportunities are represented by: a risky stock of the firm in constant net supply of 1 share that pays out dividends $\pi(t)$, $t = 1, \dots, T$; and enough zero net supply securities to dynamically complete the market. π is endogenously determined via the firm's optimization problem. Dynamic market completeness allows the construction of a unique system of Arrow-Debreu securities consistent with no arbitrage. Accordingly, we may define the *state price density process* ξ (or the pricing kernel $\xi(s)/\xi(t),\ s \geq t$), where $\xi(t, \omega)$ is interpreted as the Arrow-Debreu price (per unit probability

[5] See also DeMarzo and Bizer (1993), who were the first to point out the connection between durable goods and securities markets. DeMarzo and Urosevic (2001) also provide a comprehensive review and classification of the literature.

$\mathcal{P}$) of a unit of consumption good in state $\omega \in \Omega$ at time t. The process ξ is to be determined in equilibrium. The time-t (cum-dividend) value $V(t)$ of the stock (of the firm) is then given by

$$V(t) = E\left[\sum_{s=t}^{T} \frac{\xi(s)}{\xi(t)}\,\pi(s)\,\Big|\mathcal{F}_t\right]. \tag{1}$$

As evident from the subsequent analysis, our main results are valid in a deterministic version of our model, where ξ has only one value in each period. We introduce uncertainty to make our formulation comparable to financial markets models, standard in the literature.

2.1 The Consumer's Preferences and Optimization Problem

A representative consumer-investor-worker is endowed at time 0 with 1 share of the stock, and at each time t with $\bar{\ell}$ units of available labor, to allocate between leisure $h(t)$, and labor $\ell(t)$, for which he is paid a wage at rate $w(t)$. The consumer intertemporally chooses a non-negative consumption process c, labor process ℓ, and portfolio (of securities) process so as to maximize his lifetime utility.[6] The consumer derives a separable von Neumann-Morgenstern time-additive, state-independent utility $u(c(t)) + v(h(t))$ from consumption and leisure in $[1, T]$. The functions u and v are assumed three times continuously differentiable, strictly increasing and strictly concave, and to satisfy $\lim_{c\to 0} u'(c) = \infty$, $\lim_{h\to 0} v'(h) = \infty$.[7]

Under the complete markets assumption, the consumer-worker's dynamic optimization problem can be cast in its Arrow-Debreu formulation as a static variational problem with a single budget constraint:

$$\max_{c,\,\ell} E\left[\sum_{t=1}^{T} u(c(t)) + v(\bar{\ell} - \ell(t))\right] \tag{2}$$

$$\text{subject to} \quad E\left[\sum_{t=1}^{T} \xi(t)\Big(c(t) - w(t)\,\ell(t)\Big)\right] \le E\left[\sum_{t=1}^{T} \xi(t)\pi(t)\right]. \tag{3}$$

We do not explicitly apply the nonnegativity constraints $c(t) \ge 0$, $\ell(t) \le \bar{\ell}$, $\ell(t) \ge 0$, because the conditions $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{h\to 0} v'(h) = \infty$

[6] We introduce labor as one of the simplest, most commonly-adopted choices of input to the firm's production function; employing labor does not introduce the intertemporal complexity of employing capital.

[7] We make the assumption of separable utility for simplicity, given our intention is to focus on a comparison of monopolistic behavior in a particular good market. Our analysis readily extends to the general non-separable case $u(c,h)$. Our major comparative statics results (Propositions 2 and 4) remain valid under the assumption $u_{ch} \ge 0$ (which includes separable utility and Cobb-Douglas $u(c,h) = \frac{1}{\gamma}(c^{\rho}h^{1-\rho})^{\gamma}$, for $\gamma \in (0,1)$.)

guarantee $c(t) > 0$, $\ell(t) < \bar{\ell}$, while (in the equilibrium provided) the firm's production technology (Section 2.2) guarantees $\ell(t) \geq 0$.

The first-order conditions of the static problem (2)–(3) are

$$u'(c(t)) = y\,\xi(t)\ , \tag{4}$$

$$v'(\bar{\ell} - \ell(t)) = y\,\xi(t)\,w(t)\ , \tag{5}$$

where the Lagrangian multiplier y satisfies

$$E\left[\sum_{t=1}^{T} \xi(t)\left(c(t) - w(t)\,\ell(t)\right)\right] = E\left[\sum_{t=1}^{T} \xi(t)\pi(t)\right]\ . \tag{6}$$

Consequently,

$$\frac{v'(\bar{\ell} - \ell(t))}{u'(c(t))} = w(t)\ . \tag{7}$$

2.2 The Firm's Production and Optimization

The representative firm in this economy faces the same information structure and set of securities as the consumer. At each time $t = 1, \ldots, T$, the firm uses labor, $\ell^D(t)$, as its only input to a production technology, f, which provides consumption good as output.[8] The technology is stochastic, driven by a shock process ε, assumed (without loss of generality) to be strictly positive. The firm's output at time t is given by $f(\ell^D(t), \varepsilon(t))$. We assume f is increasing and strictly concave in its first argument and that $\lim_{\ell^D \to 0} f_\ell(\ell^D, \varepsilon) = \infty$ and $\lim_{\ell^D \to 0} f(\ell^D, \varepsilon) \geq 0$. The firm pays out a wage $w(t)$ for each unit of labor it utilizes, so its time-t profit is

$$\pi(t) = f(\ell^D(t), \varepsilon(t)) - w(t)\,\ell^D(t), \tag{8}$$

all of which it pays out as dividends to its shareholders. The firm's objective is to maximize its market value, or the present value of its expected profits under various market structures.

The firm's behavior is the main focus of this work. For the purpose of comparison, we first consider the optimal choice of a competitive firm, and the resulting competitive equilibrium (Section 2.3), and then turn to an economy where a firm exercises monopoly power in the market for the consumption good (Section 3). In the latter set-up our assumption of the firm's maximizing its value is prone to the criticism that applies to all equilibrium models with imperfect competition. To this day, the issue whether value maximization is the appropriate objective is still open (see Remark 1). Our viewpoint in this paper is to simply adopt the most well-understood equilibrium concept,

[8] For simplicity, we do not model the time-0 consumption/leisure and production choice. All our results for $t = 1, \ldots, T$ quantities, in the propositions of the paper, would remain valid if the time-0 choice were additionally modeled.

the Cournot-Walrasian equilibrium, and despite its possible criticisms, focus on the implications. In particular, we interpret our representative consumer-shareholder as a standard Walrasian price-taking agent. As such, he perceives his decisions as having no effect on the valuation of his consumption/labor stream, given by the left-hand side of the budget constraint (3). On the other hand, as a shareholder, he can affect the quantity on the right-hand side of (3), his initial wealth, which is equal to the value of the firm. Monotonicity of his indirect utility of wealth, then, justifies the firm's value maximization as a desired objective of its shareholder.

2.3 Benchmark Competitive Equilibrium

The objective of the competitive firm is choose the input ℓ^D so as to maximize its time-0 value, $V(0)$, *taking the state prices and wage as given.* The first-order conditions for a competitive firm's problem are given by

$$f_\ell(\ell^D(t),\, \varepsilon(t)) - w(t) = 0 \qquad t = 1, \dots, T. \tag{9}$$

Definition 1 (Competitive Equilibrium). *An equilibrium in an economy of one competitive firm and one representative consumer-worker is a set of prices (ξ^c, w^c) and choices (c^c, ℓ^c, ℓ^{D^c}) such that (i) the consumer chooses his optimal consumption/labor policy given the state price and wage processes, (ii) the firm makes its optimal labor choice given the state prices and wage, and (iii) the consumption and labor markets clear at all times:*

$$c^c(t) = f(\ell^{D^c}(t), \varepsilon(t)) \quad \textit{and} \quad \ell^c(t) = \ell^{D^c}(t)\,. \tag{10}$$

It is straightforward to show from (7)–(10) that in the competitive equilibrium, the equilibrium labor $\ell^c = \ell^{D^c}$ is given by

$$f_\ell(\ell^c(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \ell^c(t))}{u'(f(\ell^c(t), \varepsilon(t)))} = 0 \tag{11}$$

and the equilibrium consumption and profit processes by

$$c^c(t) = f(\ell^c(t), \varepsilon(t))\,, \quad \pi^c(t) = f(\ell^c(t), \varepsilon(t)) - f_\ell(\ell^c(t), \varepsilon(t))\, \ell^c(t)\,. \tag{12}$$

The equilibrium state price density, wage and firm value processes are given by

$$\xi^c(t) = u'(f(\ell^c(t), \varepsilon(t)))\,, \qquad w^c(t) = f_\ell(\ell^c(t), \varepsilon(t))\,, \tag{13}$$

$$V^c(t) = E\left[\sum_{s=t}^{T} \frac{u'(f(\ell^c(s), \varepsilon(s)))}{u'(f(\ell^c(t), \varepsilon(t)))} \left\{ f(\ell^c(s), \varepsilon(s)) - f_\ell(\ell^c(s), \varepsilon(s))\ell^c(s) \right\} \middle| \mathcal{F}_t \right]. \tag{14}$$

The above conditions present a fully analytical characterization of the equilibrium, with (11) determining the labor as a function of the shock ε and then (12)–(14) determining all other quantities. Equation (11) states that the marginal rate of substitution between consumption and leisure is equated to the marginal product of labor.

3 Monopolistic Equilibrium

In this section, we assume the consumption good market to be imperfect, in that the firm has monopoly power therein. We take the firm to act as a non-price-taker in its output market, taking into account the impact of its production plan choice on the price of output. The firm is still a price-taker in its input/labor market, taking the wages w as given. (The extension to the case where the firm is additionally a non-price-taker in the labor market is straightforward and is discussed in Appendix C.)

We will observe that the firm's production strategy is time-inconsistent, in the sense that a monopolist has an incentive to deviate from his time-0 plan at a later date. When the monopolist gets to time t, he no longer cares about the time-0 value of the firm; rather, he would like to revise the production plan so as to instead maximize the time-t value of the firm. In Section 3.1, we consider the optimal choice of a monopolistic firm that chooses an initial strategy so as to maximize its time-0 value and "pre-commits" to that strategy, not deviating at subsequent times. In Section 3.2, we solve for the time-consistent strategy of a monopolist who re-optimizes his production plan to maximize the firm's current value at each date t, taking into account the fact that he is not restricted from revising this plan at the future dates $s = t+1, \ldots, T$. The former can be thought of as a "long-term" strategy, providing the first-best solution to the problem of maximizing the firm's initial value. The latter can be interpreted as a "short-term" or "short-sighted" strategy since the firm continually re-optimizes every period to boost current value. We present the equilibrium for both cases.

3.1 The Pre-Commitment Case

We now formulate the monopolist's optimization problem. Recalling the price-taking consumer's demand (4), clearing in the consumption good market implies

$$\xi(t) = u'\Big(f(\ell^D(t), \varepsilon(t))\Big)/y\,. \tag{15}$$

The monopolist's influence in the good market manifests itself, via (15), as a (non-linear) impact of its input demand on state prices. Accordingly, the pre-committed monopolist solves the following optimization problem at time 0:

$$\max_{\ell^D,\,\xi} V(0) \quad \text{subject to } \xi(t) = u'\Big(f(\ell^D(t), \varepsilon(t))\Big)/y\,, \quad \forall\, t = 1, \ldots, T\,, \tag{16}$$

where y satisfies (6).

Proposition 1 presents the optimal solution to this problem, assuming it exists.[9]

[9] It is well-known that the objective function in models with monopolistic firms may not necessarily be concave and hence not satisfy the second-order

Proposition 1. *The pre-commitment monopoly optimal labor demand ℓ^D, $t = 1, \ldots, T$ satisfies*

$$f_\ell(t) - w(t) = A(t)\, f_\ell(t)\pi(t) \;>\; 0\,, \tag{17}$$

where

$$A(t) \equiv -\frac{u''(t)}{u'(t)}\,,$$

and $f(t)$, $u(t)$ and their derivatives are shorthand for $f(\ell^D(t), \varepsilon(t))$, $u(f(\ell^D(t), \varepsilon(t)))$ and their derivatives.

The structure of the first-order conditions bears resemblance to that of the textbook single-period monopolist. Any direct increase in profit due to an extra unit of input must be counterbalanced by an indirect decrease via the impact of that extra unit on the concurrent price system. In our set-up, the extent of monopoly power is driven by the current profit, the marginal product of labor and the (positive) quantity A, which captures the consumer's attitude toward risk over consumption. (The quantity A can also be restated in terms of the textbook "monopoly markup.") The higher the marginal product the more responsive is output to an extra unit of input. The more "risk-averse" the consumer, the less his consumption reacts to changes in the state price, or conversely, the more the state price reacts to changes in his consumption, and so the more incentive the monopolist has to deviate from competitive behavior. In the limit of a risk-neutral investor (preferences quasi-linear with respect to consumption), the monopolist cannot affect the state price at all and so the best he can do is behave competitively.

We now turn to an analysis of equilibrium in this economy.

Definition 2 (Monopolistic Pre-Commitment Equilibrium). *An equilibrium in an economy of one monopolistic firm and one representative consumer-worker is a set of prices (ξ^*, w^*) and choices (c^*, ℓ^*, ℓ^{D*}) such that (i) the consumer chooses his optimal consumption/labor policy given the state price and wage processes, (ii) the firm makes its optimal labor choice in (16) given the wage, and taking into account that the price system responds to clear the consumption good market, and (iii) the price system is such that the consumption and labor markets clear at all times:*

$$c^*(t) = f(\ell^{D*}(t), \varepsilon(t)) \quad \text{and} \quad \ell^*(t) = \ell^{D*}(t).$$

The fully analytical characterization of the equilibrium in the monopolistic pre-commitment economy is given by (18)–(21). Equilibrium is determined by

conditions without additional assumptions (e.g., Tirole (1988, Chapter 1)). In the pre-commitment case a sufficient condition for concavity is $u'(c)f_{\ell\ell}(\ell,\varepsilon) + 2u''(c)f_\ell(\ell,\varepsilon)(f_\ell(\ell,\varepsilon)-w)+u''(c)f_{\ell\ell}(\ell,\varepsilon)(f(\ell,\varepsilon)-w\ell)+u'''(c)f_\ell^2(\ell,\varepsilon)(f(\ell,\varepsilon)-w\ell) < 0$, $c = f(\ell,\varepsilon)$; $\forall \varepsilon$, ℓ, w. Examples of utilities and production functions that satisfy this condition include the commonly employed power preferences over consumption $u(c) = c^\gamma/\gamma$ with $\gamma \in (0,1)$ and power production $f(\ell,\varepsilon) = \varepsilon\ell^\nu$, $\nu \in (0,1)$ (no additional restriction on $v(h)$ is required).

computing the supply and demand for labor from the consumer's and firm's first-order conditions, and then applying labor market clearing $\ell^* = \ell^{D^*}$ to yield the labor as a function of the shock ε (18).[10] The remaining quantities are then straightforward to determine; we list them in (19)–(21). The equilibrium labor is given by

$$\begin{aligned} f_\ell(\ell^*(t), \varepsilon(t)) &- \frac{v'(\bar{\ell} - \ell^*(t))}{u'(f(\ell^*(t), \varepsilon(t)))} \\ &= A(t) f_\ell(\ell^*(t), \varepsilon(t)) \left[f(\ell^*(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \ell^*(t))}{u'(f(\ell^*(t), \varepsilon(t)))}\, \ell^*(t) \right] \end{aligned} \tag{18}$$

and the equilibrium consumption and profit processes by

$$c^*(t) = f(\ell^*(t), \varepsilon(t)), \quad \pi^*(t) = f(\ell^*(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \ell^*(t))}{u'(f(\ell^*(t), \varepsilon(t)))}\, \ell^*(t). \tag{19}$$

The equilibrium state price density, wage and firm value processes are given by

$$\xi^*(t) = u'(f(\ell^*(t), \varepsilon(t))), \qquad w^*(t) = \frac{v'(\bar{\ell} - \ell^*(t))}{u'(f(\ell^*(t), \varepsilon(t)))}, \tag{20}$$

$$V^*(t) = E\left[\sum_{s=t}^{T} \frac{u'(f(\ell^*(s), \varepsilon(s)))}{u'(f(\ell^*(t), \varepsilon(t)))} \left\{ f(\ell^*(s), \varepsilon(s)) - \frac{v'(\bar{\ell} - \ell^*(s))}{u'(f(\ell^*(s), \varepsilon(s)))} \ell^*(s) \right\} \Big| \mathcal{F}_t \right]. \tag{21}$$

Proposition 2 summarizes the comparison of pertinent quantities across the monopolistic and competitive economies.

Proposition 2. *The equilibrium labor, output, state price and wage in the pre-commitment monopoly economy and competitive economy, satisfy for all $t = 1, 2, \ldots T$, all $\varepsilon(t)$:*

$$\ell^*(t) < \ell^c(t), \quad f(\ell^*(t), \varepsilon(t)) < f(\ell^c(t), \varepsilon(t)), \quad c^*(t) < c^c(t) \tag{22}$$

$$\xi^*(t) > \xi^c(t), \quad w^*(t) < w^c(t). \tag{23}$$

The firm's initial value satisfies,

$$V^*(0) > V^c(0).$$

However, the time-t profit $\pi^*(t) > 0$ and firm's value $V^*(t)$ can be either higher or lower than $\pi^c(t)$ and $V^c(t)$, respectively, $\forall\, t = 1, \ldots, T$.

[10] See Lemma A.1 of Appendix A for the existence of an interior solution $\ell^*(t) \in (0, \bar{\ell})$ to equation (18).

The monopolist has an incentive to manipulate the price of consumption by producing less than his competitive counterpart. Reducing the supply raises the price at which clearing occurs, thereby increasing the valuation of the monopolist's intertemporal profits along with his stock price. The firm needs to use less input and, accordingly, the wage rate decreases. These results are consistent with the textbook static monopoly analysis. A difference in our model is that the flow of profits, $\pi(t)$, in the monopolistic equilibrium may be lower than that in the competitive benchmark (as illustrated in Example 1 of Section 3.2); this is because it is the present value of profits that our monopolist is striving to maximize, not profits per se. By restricting production, the monopolist moves away from the competitive, profit maximizing, production plan. Since the monopolist is maximizing the firm's initial value and since he has market power, it is intuitive that the firm's initial value will be higher than in the competitive case. However, we have not forced him to maximize later values of the firm, so it is not surprising that these later values may lie higher or lower than the competitive case (as illustrated in Example 1).

The optimal policies and equilibrium in this subsection rely on the ability of the monopolist to pre-commit to his time-0 plan. Without this commitment, the monopolist would later want to deviate from his initial plan; that is, at any date $t > 0$, the solution to the firm's problem

$$\max_{\ell^D(s),\, \xi(s),\, s \geq t} V(t) \qquad \text{subject to } \xi(s) = u'\Big(f(\ell^D(s), \varepsilon(s))\Big)/y\,, \qquad \forall\, s = t, \ldots, T$$

is different from his time-0 plan, unless $V(t) = 0$. Due to the intertemporal dependence of V on the price of consumption, if the monopolist solves his problem at the initial period, he would change his mind at a later stage about the optimal ξ process. In other words, his production plan is not time-consistent. A similar problem arises in the context of a durable good monopoly (e.g., Tirole (1988, Chapter 1)), or in the context of dynamic securities markets with non-price-taking investors (Basak (1997), Kihlstrom (1998)). The firm's stock in our model is similar to a durable good since it provides value over many periods. If there were a credible mechanism for the monopolist to commit to his time-0 plan, the time-0 share price of the firm that he owns would be *ceteris paribus* the highest possible; if however such commitment is impossible, we will show that the monopolist might be better off behaving competitively (Example 1). Possible pre-commitment mechanisms may include (i) employing a production technology with a time-to-build feature, and following Tirole (1988), (ii) handing the firm over to a third party instructed to implement the optimal strategy with a penalty for deviating from it (although renegotiation would be a potential issue), (iii) long-term relationships/contracts, (iv) "money-back guarantee" penalizing deviations from a targeted labor demand. Modeling additionally a pre-commitment mechanism is beyond the scope of our analysis. However, we provide the pre-commitment solution since we view it as a valuable yardstick against which we compare the time-consistent solution, reported below. Moreover, the pre-commitment solution also resembles the standard textbook monopoly behavior.

3.2 The Time-Consistent Case

We now turn to the time-consistent monopolist's optimization problem. Formally, the time-consistency requirement imposes an additional restriction on the firm's production choice: at no time t can the monopolist be willing to deviate from his time-0 strategy. If the firm is not restricted from revising its strategy at dates $t = 1, \dots, T$, it should optimally choose the intertemporal production plan taking into account that it will always act optimally in the future. We proceed to find the monopolist's subgame perfect strategy by backward induction; this is a time-consistent strategy. Specifically, the monopolist chooses the current input and price of consumption to maximize the firm's current value given the firm's future maximized value $\forall\, t = 0, \dots, T$, i.e., solves the dynamic program:

$$\begin{aligned} \overline{V}(t) &\equiv \max_{\ell^D(t),\, \xi(t)} V(t) \\ \text{subject to } \xi(t) &= u'\Big(f(\ell^D(t), \varepsilon(t))\Big)/y\,, \\ V(s) &= \overline{V}(s)\,,\ \ (\ell^D(s),\, \xi(s)) = \operatorname{argmax} V(s)\,,\ \forall\, s = t+1, \dots, T, \end{aligned} \tag{24}$$

where y satisfies (6).

Proposition 3 presents the optimal solution to this problem, assuming it exists.[11]

Proposition 3. *The time-consistent monopoly optimal labor demand* ℓ^D, $t = 1, \dots, T-1$ *satisfies*

$$f_\ell(t) - w(t) = -A(t)\, f_\ell(t)\, V_{ex}(t) \ \ < 0\,, \tag{25}$$

where $V_{ex}(t)$ *denotes the ex-dividend value of the firm given by*

$$\begin{aligned} V_{ex}(t) &\equiv E\left[\sum_{s=t+1}^{T} \frac{u'(s)}{u'(t)} [f(s) - w(s)\, \ell(s)] \,\Big|\, \mathcal{F}_t\right] \qquad \forall\, t = 1, \dots, T-1; \\ V_{ex}(T) &= 0. \end{aligned}$$

At time T, *the time-consistent monopoly optimal labor demand* ℓ^D *satisfies* $f_\ell(T) - w(T) = 0$.

[11] In the time-consistent case a sufficient condition for concavity of the objective function is $-2\,\frac{u''(c)}{u'(c)} + \frac{u'''(c)}{u''(c)} < -\frac{f_{\ell\ell}(\ell,\varepsilon)}{f_\ell^2(\ell,\varepsilon)}$, $c = f(\ell, \varepsilon)$; $\forall \varepsilon$, ℓ, satisfied, for example, by power preferences $u(c) = c^\gamma/\gamma$ with $\gamma \in [0, 1)$, which includes $u(c) = \log(c)$ (no additional restriction on the production function $f(\ell, \varepsilon)$ is required).

The ex-dividend value of the firm's stock at time T is zero; hence the terminal first-order condition and optimal choice of the firm coincide with those of the competitive firm. The first-order condition at time T is the terminal condition for the backward induction: the remaining labor choices are determined by solving (25) backwards.

Similarly to the pre-commitment case, at the optimum for the time-consistent monopolist the increase in profit due to an extra unit of input used must counteract the indirect decrease via the impact of that extra unit on the price system. However, the extent of this monopoly power is now driven by the negative of the ex-dividend stock price in place of the current profit. (The consumer's attitude toward risk for consumption and the marginal product of labor appear as in the pre-commitment case.) The short-sighted monopolist only cares about (and tries to manipulate) the current valuation of the stock, which is made up of current profit plus the ex-dividend value of the firm. Given the time-consistency restriction, this monopolist takes the future value of profits as given and can only boost the firm's ex-dividend value by depressing the current price of consumption. So, while it was in the pre-committed monopolist's interest to boost the current price of consumption, it is in the time-consistent monopolist's interest to depress it; hence the negative term in (25). The ex-dividend value of the firm appears because this is what he manipulates; the higher is $V_{ex}(t)$, the stronger the incentive to cut concurrent profit to manipulate the firm's valuation. Hence, the firm's value serves as an extra variable in determining the optimal policy of the firm.

We now define an equilibrium in a monopolistic economy containing the time-consistent firm.

Definition 3 (Monopolistic Time-Consistent Equilibrium). *An equilibrium in an economy of one monopolistic firm and one representative consumer-worker is a set of prices $(\hat{\xi}, \hat{w})$ and choices $(\hat{c}, \hat{\ell}, \hat{\ell}^D)$ such that (i) the consumer chooses his optimal consumption/labor policy given the state price and wage processes, (ii) the firm chooses a time-consistent strategy so as to maximize its objective in (24) given the wage, and taking into account that the price system responds to clear the consumption good market, and (iii) the consumption and labor markets clear at all times:*

$$\hat{c}(t) = f(\hat{\ell}^D(t), \varepsilon(t)) \quad \textit{and} \quad \hat{\ell}(t) = \hat{\ell}^D(t).$$

The fully analytical characterization of the monopolistic time-consistent equilibrium is presented in (26)–(29). Again, from labor market clearing, we first determine the labor as a function of the shock ε (26).[12] Then, (27)–(29) give the remaining equilibrium quantities. The equilibrium labor is given by

$$f_\ell(\hat{\ell}(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \hat{\ell}(t))}{u'(f(\hat{\ell}(t), \varepsilon(t)))} = -A(t)\, f_\ell(\hat{\ell}(t), \varepsilon(t))\, V_{ex}(t) \tag{26}$$

[12] See Lemma A.3 of the Appendix for the existence of an interior solution $\hat{\ell}(t) \in (0, \bar{\ell})$ to equation (18).

and the equilibrium consumption and profit processes by

$$\hat{c}(t) = f(\hat{\ell}(t), \varepsilon(t)) \,, \qquad \hat{\pi}(t) = f(\hat{\ell}(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \hat{\ell}(t))}{u'(f(\hat{\ell}(t), \varepsilon(t)))}\,\hat{\ell}(t) \,. \quad (27)$$

The equilibrium state price density, wage and firm value processes are given by

$$\hat{\xi}(t) = u'(f(\hat{\ell}(t), \varepsilon(t))) \,, \qquad \hat{w}(t) = \frac{v'(\bar{\ell} - \hat{\ell}(t))}{u'(f(\hat{\ell}(t), \varepsilon(t)))} \,, \quad (28)$$

$$\hat{V}(t) = E\left[\sum_{s=t}^{T} \frac{u'(f(\hat{\ell}(s), \varepsilon(s)))}{u'(f(\hat{\ell}(t), \varepsilon(t)))} \left\{ f(\hat{\ell}(s), \varepsilon(s)) - \frac{v'(\bar{\ell} - \hat{\ell}(s))}{u'(f(\hat{\ell}(s), \varepsilon(s)))}\hat{\ell}(s) \right\} \Big| \mathcal{F}_t \right] (29)$$

Proposition 4 presents a comparison of the monopolistic time-consistent and the competitive economies.

Proposition 4. *The equilibrium labor, output, state price and profit in the time-consistent monopolistic and the competitive economies satisfy, for all* $t = 1, \dots T-1$, *all* $\varepsilon(t)$:

$$\hat{\ell}(t) > \ell^c(t), \quad f(\hat{\ell}(t), \varepsilon(t)) > f(\ell^c(t), \varepsilon(t)), \quad \hat{c}(t) > c^c(t) \quad (30)$$

$$\hat{\xi}(t) < \xi^c(t), \quad \hat{w}(t) > w^c(t), \quad \hat{\xi}(t)\hat{\pi}(t) < \xi^c(t)\pi^c(t), \quad \hat{\pi}(t) < \pi^c(t). \quad (31)$$

The firm's initial value satisfies,

$$\hat{V}(0) < V^c(0).$$

At time T, *all the equilibrium quantities coincide with those of the competitive benchmark. The firm's value,* $\hat{V}(t)$, *can be either higher or lower than in the competitive economy,* $V^c(t)$, $t = 0, \dots, T-1$.

The short-sighted monopolist's behavior is exactly opposite to that of the monopolist who pre-commits. He desires to depress the current price of consumption (to boost the current stock price); hence the lower level of state price density. He achieves this by increasing output, hence demanding more labor input, and in return pushing up the wage rate. Although this monopolist moves his labor demand in the opposite direction to the pre-committed monopolist, he also moves away from the maximum profit point and so profits are reduced relative to the competitive case. In his attempt to manipulate the stock price, concurrent profit and its value are cut. This type of behavior may overall result in a higher or lower value of the firm's stock price. By consistently reducing the value of concurrent profit, the short-sighted owners of the monopolistic firm may, then, cause damage to the firm and its stock price. This is so even though the time-consistent monopolist is restricted to maximize the firm's value at each point in time. For the pre-committed monopolist, we could explain this result by his not attempting to maximize this quantity, but here the explanation must be more complex (see Examples 1 and 2).

We now present three examples which deliver additional insights into the equilibrium quantities. The first example demonstrates that the monopolistic firm's profits can go negative in equilibrium. The second example allows an investigation of the relationship between the monopolistic and the competitive firm values, as well as the infinite horizon limit. The third example shows how under an alternative, albeit extreme, form of a production opportunity the time-consistent monopolist's profits go to zero, while the pre-committed monopolist's profits do not.

Example 1 (Negative Profits) Here, we provide a numerical example in which the time-consistent monopolistic firm's equilibrium profits $\hat{\pi}(t)$ are sometimes negative, and its stock price is always lower than that of a competitive firm. Consider the following parameterization:

$$u(c) + v(\bar{\ell} - \ell) = \frac{c^\gamma}{\gamma} + \frac{(\bar{\ell} - \ell)^\rho}{\rho}, \ \gamma \in (0,1), \ \rho < 1;$$
$$f(\ell, \varepsilon) = \varepsilon + \ell^\nu, \ \nu \in (0,1); \quad T = 2.$$

The productivity shock enters the technology additively; this type of shock can be interpreted as providing an additional endowment as in a Lucas (1978)-type pure-exchange economy. We set $\bar{\ell} = 1, \ \gamma = 0.5, \ \rho = \nu = 0.9, \ \varepsilon(1) = 0.1, \ \varepsilon(2) = 0.5$ (deterministic for simplicity of exposition). The resulting equilibrium labor choices, profits and stock prices are reported in Table 1.

Table 1. Equilibrium labor choice, profits and firm value in the competitive, monopolistic pre-commitment and monopolistic time-consistent economies. The reported values are for the economies parameterized by $u(c) + v(\bar{\ell} - \ell) = c^\gamma/\gamma + (\bar{\ell} - \ell)^\rho/\rho, \ f(\ell, \varepsilon) = \varepsilon + \ell^\nu, \ T = 2$, with $\bar{\ell} = 1, \ \gamma = 0.5, \ \rho = \nu = 0.9, \ \varepsilon(1) = 0.1, \ \varepsilon(2) = 0.5$

	Competitive Equilibrium	Monopolistic Pre-Commitment Equilibrium	Monopolistic Time-Consistent Equilibrium
$\ell(1)$	0.60	0.41	0.84
$\ell(2)$	0.37	0.01	0.37
$\pi(1)$	0.16	0.22	-0.27
$\pi(2)$	0.54	0.51	0.54
$V(0)$	0.75	1.02	0.54
$V(1)$	0.65	0.75	0.52
$V(2)$	0.54	0.51	0.54

At time 1 the time-consistent monopolist sacrifices his concurrent profit in order to boost the time-1 value of the stock. Indeed, consistent with Proposition 4, depending on the parameterization, nothing prevents the sign of $\hat{\pi}(1)$ from becoming negative. This is a notable feature of our solution. Although real-world firms frequently announce losses, this phenomenon cannot be generated in a competitive model where production occurs within the decision period (provided the firm has an option to costlessly shut down during that period). Negative profits can obtain in models of predation, where rivalrous producers are competing for market share (e.g., Tirole (1988, Chapter 9)); in our model, in contrast, the monopolist controls the entire market and does not fear entry, yet his profit may still go negative.

A further striking conclusion revealed by Table 1 is that non-competitive behavior of the firm does not necessarily result in a higher value of the firm; in fact, the time-consistent monopolistic firm's stock price is lower than its competitive counterpart at all times. The recognition of his monopoly power actually damages the time-consistent monopolist at all points in time, and he would be better off behaving competitively. For completeness and comparison we also provide the equilibrium quantities for the monopolistic pre-commitment economy. Example 2, having a longer time horizon, and yielding an explicit solution, allows us to investigate this stock price effect more closely.

Example 2 (Firm Value Comparison and Time Horizon Effects)
Since it appears that the intertemporal nature of the problem is critical, in this example we extend the horizon beyond $T = 2$. It would also be important to discuss the limiting case of our model as $T \to \infty$, since in some models (e.g., repeated principal-agent), the equilibrium changes dramatically on transitioning from a finite to infinite horizon. In order to obtain explicit formulae for all equilibrium quantities, we simplify beyond the general specification of the utility function employed so far. The utility we adopt has the form $u(c(t), t) - v(\ell(t), t) = \beta^t(\log c(t) - \ell(t)^\rho/\rho),\ \beta \in (0,1),\ \rho > 1$, where β is the time discount factor, required to make pertinent quantities well-defined in the limiting case of $T \to \infty$, and the second term is the disutility of labor. We show equilibrium labor to still be bounded from above by $\bar{\ell} = 1$. The firm's technology is represented by $f(\ell(t), \varepsilon(t)) = \varepsilon(t)\,\ell(t)^\nu,\ \nu \in (0,1)$. To ensure that the equilibrium quantities are well-defined in the infinite horizon limit, we assume that ε is bounded, as well as assume away speculative bubbles. The resulting exact formulae for the equilibrium labor choices, profits and stock prices are reported in Table 2, with the comparative statics as in Proposition 4.[13] The time-consistent labor choice monotonically increases as $T - t$ increases, converging to a limit (and diverging away from the competitive labor, which is constant). As $T - t$ increases, time-consistent profits fall,

[13]We do not report the monopolistic pre-commitment equilibrium quantities since for logarithmic utility over consumption, the monopolist's problem is not well-defined, as is the case in the standard textbook treatment of monopoly (Mas-Colell, Whinston and Green (1995, p.429)). Moreover, one may extend our earlier analysis of the monopolistic pre-commitment and competitive equilibria to the case of infinite horizon without major difficulties.

although do not vanish. At this point, the parallel with to the Coase (1972) conjecture is unavoidable: *ceteris paribus*, the more future production decisions the time-consistent firm can make, the lower its profits today. Does the firm erode its profits by allowing for more future decisions? We revisit this issue in Section 4.

Table 2. Equilibrium labor choice, profits, and firm value and their limits as $T - t \to \infty$ in the competitive and monopolistic time-consistent economies. The reported formulae are for the economies parameterized by $u(c(t), t) - v(\ell(t), t) = \beta^t(\log c(t) - \ell(t)^\rho/\rho)$, $\beta \in (0,1)$, $\rho > 1$; $f(\ell(t), \varepsilon(t)) = \varepsilon(t)\,\ell(t)^\nu$, $\nu \in (0,1)$

	Competitive Equilibrium	Monopolistic Time-Consistent Equilibrium
$\ell(t)$	$\nu^{1/\rho}$	$\left[\nu \frac{1-(\beta(1-\nu))^{T-t+1}}{1-\beta(1-\nu)}\right]^{1/\rho}$
$\pi(t)$	$\varepsilon(t)\ \nu^{\nu/\rho}\,(1-\nu)$	$\varepsilon(t)\ \left[\nu \frac{1-(\beta(1-\nu))^{T-t+1}}{1-\beta(1-\nu)}\right]^{\nu/\rho} \times \frac{(1-\nu)(1-\beta)+\nu(\beta(1-\nu))^{T-t+1}}{1-\beta(1-\nu)}$
$V(t)$	$\varepsilon(t)\ \nu^{\nu/\rho}\,(1-\nu) \times \frac{1-\beta^{T-t+1}}{1-\beta}$	$\varepsilon(t)\ \nu^{\nu/\rho}\,(1-\nu)\ \left[\frac{1-(\beta(1-\nu))^{T-t+1}}{1-\beta(1-\nu)}\right]^{\nu/\rho+1}$
$\lim_{T-t\to\infty} \ell(t)$	$\nu^{1/\rho}$	$\left[\frac{\nu}{1-\beta(1-\nu)}\right]^{1/\rho}$
$\lim_{T-t\to\infty} \pi(t)$	$\varepsilon(t)\ \nu^{\nu/\rho}\,(1-\nu)$	$\varepsilon(t)\ \left[\frac{\nu}{1-\beta(1-\nu)}\right]^{\nu/\rho}\ \frac{(1-\nu)(1-\beta)}{1-\beta(1-\nu)}$
$\lim_{T-t\to\infty} V(t)$	$\frac{\varepsilon(t)\ \nu^{\nu/\rho}\,(1-\nu)}{1-\beta}$	$\frac{\varepsilon(t)\ \nu^{\nu/\rho}\,(1-\nu)}{[1-\beta(1-\nu)]^{\nu/\rho+1}}$

Of special interest is the value of the firm's stock. In Figure 1, we plot the firm value in the competitive and monopolistic equilibria as a function of time. The monopolistic firm value can be either higher or lower than that of the competitive firm, depending on the age of the firm. At first blush, this result seems surprising, because the time-consistent monopolist is optimizing the firm's value at each point in time. Since he has more market power than a competitive firm, how can his firm's value come out lower? At time t he solves the problem

$$\hat{V}(t) = \max_{\ell^D(t),\xi(t)} f(\ell^D(t), \varepsilon(t)) - \hat{w}(t)\ell^D(t) + \frac{1}{u'(f(\ell^D(t), \varepsilon(t)))}\, E\left[\hat{\xi}(t+1)\,\hat{V}(t+1)\,|\mathcal{F}_t\right].$$

The competitive firm solves the same problem except that: it has no power over $\xi(t)$; and that $\hat{w}(t)$, $\hat{V}(t+1)$, $\hat{\xi}(t+1)$ are replaced by their respective values in the competitive equilibrium. This latter distinction explains why the monopolist's firm value may come out lower: he faces both a different equilibrium wage and different choices of the future state prices and firm value.

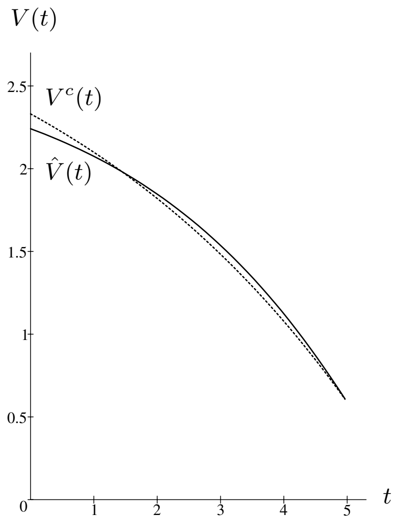

Fig. 1. Equilibrium firm value versus time in the competitive and monopolistic time-consistent economies. The dotted plot is for the competitive economy and the solid plot is for the monopolistic time-consistent. The economies are parameterized by $u(c(t), t) - v(\ell(t), t) = \beta^t(\log c(t) - \ell(t)^\rho/\rho)$, $\beta = 0.9$, $\rho = 1.05$; $f(\ell(t), \varepsilon(t)) = \varepsilon(t)\,\ell(t)^\nu$, $\nu = 0.2$, $\varepsilon(t) = 1$, $\forall t$; $T = 5$

Figure 1 also reveals that in this example the monopolistic firm increases the firm's value later in its lifetime and decreases it earlier in life. This is because the monopolistic firm competes with itself in future time periods; earlier in life it faces more competition, which adversely affects its value. In the limit of $T - t \to \infty$, the competitive firm value is unambiguously higher (Table 2).

Example 3 (Constant Returns to Scale Technology) A constant returns to scale technology is widely employed in both the monopoly literature (e.g., Tirole (1988)), and asset pricing literature (e.g., the workhorse production asset pricing model of Cox, Ingersoll and Ross (1985)). Thus far, we have been assuming strictly decreasing returns to labor; in this example, we extend our analysis to the case of constant returns, $f(\ell, \varepsilon) = \varepsilon\ell$. The firm's profit is now given by $\pi(t) = \varepsilon(t)\ell(t) - w(t)\ell(t)$.

For the competitive firm to demand a finite positive amount of labor, the equilibrium wage $w^c(t)$ must equal $\varepsilon(t)$; the equilibrium labor $\ell^c(t)$ is then determined from the consumer's optimization (7):

$$\frac{v'(\bar{\ell} - \ell^c(t))}{u'(f(\varepsilon(t), \ell^c(t)))} = \varepsilon(t). \tag{32}$$

The remaining equilibrium quantities, for all $t = 1, \dots, T$, are

$$c^c(t) = f(\ell^c(t), \varepsilon(t)), \quad \pi^c(t) = 0,$$
$$\xi^c(t) = u'(f(\ell^c(t), \varepsilon(t)), \quad V^c(0) = V^c(t) = 0.$$

Adaptation of our analysis in Section 3.1 shows that the pre-commitment monopolistic equilibrium labor demand $\ell^*(t)$ (assuming it exists and yields a maximum in the firm's problem) solves

$$-\frac{u''(f(\ell^*(t), \varepsilon(t)))}{u'(f(\ell^*(t), \varepsilon(t)))}\varepsilon(t)\ell^*(t) = 1\,, \tag{33}$$

implying $\ell^*(t) < \ell^c(t)$. The remaining equilibrium quantities are obtained from (19)–(21). The pre-commitment equilibrium, then, retains the main implications derived in Section 3.1; in particular, the firm cuts production and raises the price of output. The main difference from Section 3.1 is that monopoly profits are now always higher than the competitive ones (which are zero), consistent with the prediction of the textbook monopoly model. The firm's value is also higher.

The monopolistic time-consistent equilibrium coincides with the competitive one. To see why, recall from Proposition 4, that at the final time, $\hat{\pi}(T) = \pi^c(T)$. Since $\pi^c(T) = 0$, the time-(T-1) ex-dividend value of the monopolistic firm, $\hat{V}_{ex}(T-1)$, is zero; hence by backward induction it can be shown that $\hat{\ell}(t) = \ell^c(t)\ \forall t$, and the competitive equilibrium obtains:

$$\hat{c}(t) = f(\ell^c(t), \varepsilon(t)), \quad \hat{\pi}(t) = 0,$$
$$\hat{\xi}(t) = u'(f(\ell^c(t), \varepsilon(t)), \quad \hat{V}(0) = \hat{V}(t) = 0, \quad t = 1, \dots T.$$

In contrast to the decreasing returns to scale case, the time-consistent monopolist loses all his monopoly power. Accordingly, the value of his stock, as well as profit, are strictly lower than in the pre-commitment equilibrium. It is quite clear that the short-sighted behavior has a devastating effect on the firm's valuation. Ironically, it benefits the representative consumer, who, in effect, sets the labor demand of the firm so as to maximize his own expected lifetime utility.

Remark 1 (Price Normalization in Monopolistic Models).

Ours is a general equilibrium model with imperfect competition; accordingly, it is not immune to the general criticism of all such models initiated by Gabszewicz and Vial (1972) (see also Mas-Colell (1982)). The criticism has to do with the so-called "price-normalization problem" which arises in the presence of multiple goods because Walrasian (and Cournot-Walrasian) equilibrium theory determines relative prices but has little to say about nominal price formation. In the perfectly competitive benchmark, this deficiency is not an issue, since the nominal price indeterminacy does not affect real quantities. In the monopolistic (imperfectly competitive) equilibrium, however, the real quantities become dependent on the choice of price normalization. Indeed, for some choices of normalization, Dierker and Grodal (1986) show that no equilibrium exists, while for others it does exist. This prompted us to consider the robustness of our results to alternative specifications of the numeraire. Our model contains two commodities, the consumption good and labor, at each time and state. Instead of the consumption good, we could have specified a basket of commodities consisting of α units of the consumption good and $(1-\alpha)$ units of labor as the numeraire, $\alpha \in (0,1)$.[14] Our main qualitative results and corresponding intuitions would then still remain valid, as demonstrated in Appendix B.

Related to the price-normalization problem, is the issue of whether maximizing the firm's value through time is the correct objective. From the point-of-view of realism, it is generally accepted that this is what a firm should do, so the problem we solve is consistent with conventional thinking. However, the literature such as Dierker and Grodal (1986) (see also Dierker and Grodal (1999)), clearly casts some doubt on this approach. Hart (1985) suggests that profit maximization may be inappropriate in a monopolistic world because "the owners of the firm are interested not in monetary profit per se but in what that profit can buy". Although Hart discusses maximization of owner's utility as an alternative objective, he points out that the question of how to aggregate is not resolved when there are multiple owners. Similarly, Bonanno (1990) (in his survey) remarks, "unfortunately we are still far from a satisfactory theory of general equilibrium with imperfect competition". To this day, the price normalization and the objective of the firm under imperfect competition in general equilibrium remain open issues. Based on this, we have simply adopted the most commonly employed price normalization, as well as the most conventionally considered objective function as the best starting point, and we focus on the implications.

[14] Such a basket is a natural and realistic numeraire to adopt, as argued, for example, by Pavlova and Rigobon (2003) (for more discussion of prices indexes, see Schultze (2003)).

4 Connection to the Coase Conjecture

As we discussed in Section 3, the time-inconsistency of the firm's production plan in our monopolistic pre-commitment equilibrium is similar to the time-inconsistency arising in the context of a durable good monopoly (e.g., Tirole (1988, Chapter 1) and references therein). The long-lived firm's stock in our model is similar to a durable good in that its value is durable over many periods. The durability of goods implies that a monopolistic firm in a sense competes with itself in pricing goods that it produces at other times and must take this into account in its production behavior. Similarly, in our context, "current owners" of the firm can be thought of as competing with "future owners" in maximizing the value of the firm's stock; they must account for possible revisions to today's intertemporal production plan by the future owners. This effective competition between past and future sales in the durable good problem tends to weaken the monopoly power and may ultimately enforce competitive behavior. This observation is the Coase (1972) conjecture: as the time interval between successive decision points shorten, the price of the durable good charged by the monopolist converges to the competitive price (marginal cost), and the monopoly profits are driven to zero.

To examine the extension of Coase's intuition to the case of a long-lived monopolistic firm manipulating the price of its stock, we explore the limiting case of our economy as the decision-making interval shrinks.[15] We partition the time horizon $[0,T]$ in our economy into n small intervals of length Δ, so that $t = 0, \Delta, 2\Delta, \ldots, n\Delta = T$. All flow variables in the economy ($\ell(t)$, $c(t)$, $f(t)$ and $\pi(t)$) are now interpreted as "flows over the interval t to $t+\Delta$". We then take the limit as $n \to \infty$ ($\Delta \to 0$). Proposition 5 reports the resulting optimality conditions and equilibrium characterization for the competitive, monopolistic pre-commitment and monopolistic time-consistent economies, assuming appropriate regularity conditions are satisfied.

Proposition 5. *In the continuous-time limit, as $n \to \infty$, $\forall t$:*

(i) The competitive firm's optimal labor demand ℓ^D satisfies

$$f_\ell(\ell^D(t), \varepsilon(t)) - w(t) = 0. \tag{34}$$

Consequently, the competitive equilibrium characterization is given by the conti-nuous-time analogs of equations (11)–(14).

[15] Our goal here is to merely explore what happens to equilibrium quantities as the firm's decisions become more frequent, and to not present a continuous-time extension of our model. Similar discrete-time models with $T = \infty$ have been argued to exhibit multiple equilibria as the decision-making interval shrinks, according to the Folk Theorem (see also Ausubel and Deneckere (1989)), which motivated us to keep the horizon T finite.

(ii) The pre-commitment monopolistic firm's optimal labor demand ℓ^D *satisfies*

$$f_\ell(\ell^D(t), \varepsilon(t)) - w(t) = A(t)\, f_\ell(t)\, \pi(t) \ > 0. \tag{35}$$

Consequently, the monopolistic pre-commitment equilibrium characterization is given by the continuous-time analogs of equations (18)–(21).

(iii) Assume further that $\varepsilon(t) \in [k, K]$, $0 < k < K < \infty$. *Then the continuous-time limit exists for the monopolistic time-consistent equilibrium. The time-consistent monopolistic firm's optimal labor demand* ℓ^D *satisfies*

$$\pi(\ell^D(t), \varepsilon(t)) = f(\ell^D(t), \varepsilon(t)) - w(t)\ell^D(t) = 0, \qquad \forall t. \tag{36}$$

Consequently, the equilibrium labor for $\forall t$ *is given by*

$$f(\hat{\ell}(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \hat{\ell}(t))}{u'(f(\hat{\ell}(t), \varepsilon(t)))}\, \hat{\ell}(t) = 0. \tag{37}$$

Furthermore, for $\forall t$,

$$\hat{c}(t) = f(\hat{\ell}(t), \varepsilon(t)), \quad \hat{\pi}(t) = 0, \quad \hat{\xi}(t) = u'(f(\hat{\ell}(t), \varepsilon(t)),$$

$$\hat{w}(t) = \frac{v'(\bar{\ell} - \hat{\ell}(t))}{u'(f(\hat{\ell}, \varepsilon(t)))} \quad \textit{and} \quad \hat{V}(t) = 0.$$

The equilibrium characterization of the competitive and the monopolistic pre-commitment economies are exactly analogous to the discrete-time characterizations. Our main focus is on the monopolistic time-consistent equilibrium. There, the firm's profit and value indeed shrink to zero, as the monopolist's decision interval becomes arbitrarily small – in line with the Coase conjecture. In other words, the competing forces between current profits and the valuation of future profits may diminish shareholder value, fully destroying it in the continuous-time limit. Note, however, that our monopolist's profits are not equated to the competitive ones. This is simply because, in contrast to the textbook Marshallian framework adopted by Coase where competition implies free entry of firms, competitive behavior in our setting does not yield zero profits in equilibrium. This distinction is due to our Arrow-Debreu-McKenzie setting; in particular, to the assumptions that there is a fixed number of firms, and that the technology of each firm is convex.

5 Conclusion

In this paper, we model a production economy which, along with a standard representative consumer, includes a large value-maximizing monopolistic firm.

The firm manipulates its valuation as well as the price of the good that it produces. This feature makes its time-0 production plan time-inconsistent. We address the time consistency problem in two polar ways: first, we assume the firm can credibly commit to never revoking its time-0 decision (long-term policy); and second, we assume that the firm takes into account that it will revise its production plan at each decision point (short-term policy). At a general equilibrium level, we show that the long-term policy is largely consistent with the implications of the textbook static monopoly model; as compared to the competitive economy, the output is decreased and the price of consumption is increased, yet the profits and the firm's value can be either increased or decreased. The short-term policy, however, is at odds with the static model; the output is increased while the price is decreased. More strikingly, under the short-term policy, the profits in every period are decreased, and may even go negative, while the firm's value can drop below than in the competitive benchmark. The distinction between the long- and short-term policies becomes even sharper in the continuous-time limit of our economy: while the pre-commitment equilibrium retains its basic discrete-time structure and implications, the time-consistent equilibrium tends to the limit of zero profits and hence zero firm's value at all times. Since our primary focus is comparison between the long- and short-term approaches, and comparison with the textbook model, we have omitted an analysis of the volatility and risk premium of the monopolistic firm's stock price, and of the impact of monopoly power on the equilibrium interest rate, market price of risk, output and consumption variabilities. This analysis would be straightforward in the continuous-time limit of our economy, but less tractable in discrete-time.

Our analysis involving a single monopolistic firm, a single consumption good, and a representative consumer is, admittedly, simplistic. Our goal in this paper has been to develop the minimal setting possible capturing the mechanism through which the firm's market power may impact valuation in the economy, and not to produce the most empirically plausible model. Throughout the paper, we also discuss the robustness of our implications to alternative modeling approaches. For realism, however, one would need to extend this work to include multiple imperfectly competitive firms producing homogeneous, or differentiated, goods, or multiple consumers. The former (multiple firms) would draw from the results of the oligopoly literature, while the latter (multiple consumers) would involve aggregation of the consumers' preferences into a representative agent.

Appendix A. Proofs

Proof of Proposition 1.

Since there is no consumption or production at time 0, we can set $\xi(0) = 1/y$ w.l.o.g. Upon substitution of (15) into (1), and using the definition (8), we obtain an equivalent representation of (16):

$$\max_{\ell} E\left[\sum_{t=1}^{T} u'(f(\ell(t),\varepsilon(t))\Big\{f(\ell(t),\varepsilon(t)) - w(t)\ell(t)\Big\}\right], \tag{A.1}$$

where the firm's objective is now a function of $\{\ell(t);\ t = 1,\ldots,T\}$ only. The first-order conditions for (A.1) are given by

$$u''(f(\ell(t),\varepsilon(t)))\, f_\ell(\ell(t),\varepsilon(t))(f(\ell(t),\varepsilon(t)) - w(t)\ell(t)) \\ + u'(f(\ell(t),\varepsilon(t)))\,(f_\ell(\ell(t),\varepsilon(t)) - w(t)) = 0, \quad \forall t = 1,\ldots,T.$$

Using the definition of $A(t)$ and rearranging above yields (17). The sufficient condition for concavity in footnote 9 guarantees that the labor solving (17) is the maximizer for (16). Since the firm has the option to shut down (set $\ell^D(t) = 0$) during any period $[t, t+1]$ in this static optimization, $\pi(t) \geq 0$ and $u'(f(\ell(t),\varepsilon(t))\pi(t) \geq 0$. This together with strict concavity of each term in (A.1) (by assumption) guarantees that $u'(f(\ell^D(t),\varepsilon(t))\pi(t) > 0$ at the optimum, and hence $\pi(t) > 0$. $\pi(t) > 0$ together with $A(t) > 0$, $f_\ell(\ell,\varepsilon(t)) > 0$ (by assumptions on preferences and technology), ensures that the expression on the right-hand side of (17) is strictly positive. □

The following Lemmas A.1 and A.2 are employed in the proofs below. Lemma A.1 shows that under regularity conditions (satisfied, for example, by power preferences over consumption $u(c) = c^\gamma/\gamma$ with $\gamma \in (0,1)$ and power production $f(\ell,\varepsilon) = \varepsilon\ell^\nu$, $\nu \in (0,1)$; no additional restriction on $v(h)$ is required) both equilibrium ℓ^c and ℓ^* belong to the interior of $[0,\bar{\ell}]$. Lemma A.2 is employed in the proofs of Propositions 2, 4 and 5.

Lemma A.1.

Under the standard assumptions on preferences and production (see Sections 2.1 and 2.2), there exists a unique solution, $\ell^c(t) \in (0,\bar{\ell})$, to (11). Assume further that $\lim_{\ell\to 0} u'(f(\ell,\varepsilon))\, f(\ell,\varepsilon)) < \infty$, $\lim_{\ell\to 0} u'(f(\ell,\varepsilon))\,\ell < \infty$, $\lim_{\ell\to 0} -u''(f(\ell,\varepsilon))\ f(\ell,\varepsilon)/u'(f(\ell,\varepsilon)) < 1$ *and* $\lim_{\ell\to 0}(u''(f(\ell,\varepsilon)))\, f_\ell(\ell,\varepsilon)\,\ell + u'(f(\ell,\varepsilon)) < \infty\ \forall\varepsilon$. *Then there exists a solution, $\ell^*(t) \in (0,\bar{\ell})$, to (18).*

Proof of Lemma A.1. Since $\lim_{\ell\to 0} u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon) = \infty > \lim_{\ell\to 0} v'(\bar{\ell} - \ell)$, and $\lim_{\ell\to\bar{\ell}} v'(\bar{\ell} - \ell) = \infty > \lim_{\ell\to\bar{\ell}} u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon)$, and since $u'f_\ell$ is decreasing in ℓ while v' is increasing in ℓ, there exists a unique solution, $\ell^c \in (0,\bar{\ell})$, to (11). Since $\lim_{\ell\to 0} u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon) - v'(\bar{\ell} - \ell) - A(t)f_\ell(\ell,\varepsilon)(u'(f(\ell,\varepsilon)f(\ell,\varepsilon) - v'(\bar{\ell} - \ell)\ell) = \infty$ and $\lim_{\ell\to\bar{\ell}} u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon) - v'(\bar{\ell} - \ell) - A(t)f_\ell(\ell,\varepsilon)(u'(f(\ell,\varepsilon))f(\ell,\varepsilon) - v'(\bar{\ell} - \ell)\ell) = -\infty$, continuity of $u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon) - v'(\bar{\ell} - \ell) - A(t)f_\ell(\ell,\varepsilon)(u'(f(\ell,\varepsilon))f(\ell,\varepsilon) - v'(\bar{\ell} - \ell)\ell)$ on $(0,\bar{\ell})$ together with the boundary behavior at $\ell \to 0$ and $\ell \to \bar{\ell}$ ensures existence of a solution, $\ell^* \in (0,\bar{\ell})$, to (18). □

Lemma A.2.

$f_\ell(\ell,\varepsilon) - \frac{v'(\bar{\ell}-\ell)}{u'(f(\ell,\varepsilon))}$ *is decreasing in* ℓ *for all* ε.

Proof of Lemma A.2. Assumptions on preferences and production and straightforward differentiation deliver the result. □

Proof of Proposition 2.

Since the right-hand side of (18) is strictly greater than zero due to Proposition 1, and the right-hand side of (11) is zero, it follows from Lemma A.2 that $\ell^*(t) < \ell^c(t)$. The remaining inequalities in (22)–(23) are then straightforward to derive: they are due to $f(\ell,\varepsilon)$ and $w = v'(\bar{\ell}-\ell)/u'(f(\ell,\varepsilon))$ being increasing in ℓ, $\forall\varepsilon$, the good market clearing, and $\xi = u'(f(\ell,\varepsilon))$ being decreasing in ℓ, $\forall\varepsilon$. Equation (6) is automatically satisfied in equilibrium due to clearing in the good and labor markets, hence y is indeterminate and we can normalize $y = 1$. Together with our earlier normalization $\xi(0) = 1/y$, this yields $\xi(0) = 1$ in both the competitive and monopolistic pre-commitment equilibria. For $w = w^*$, the function $\xi\pi(\ell,\varepsilon,w) \equiv u'(f(\ell,\varepsilon))(f(\ell,\varepsilon) - w\ell)$ achieves its maximum at ℓ^*. $\xi\,\pi$ is strictly decreasing in w, hence for any $w > w^*$, $u'(f(\ell,\varepsilon))(f(\ell,\varepsilon) - w\ell) < u'(f(\ell^*,\varepsilon))(f(\ell^*,\varepsilon) - w^*\ell^*)$, for all ℓ. This together with $w^c(t) > w^*(t)$ implies that for each term in the expression for $V(0)$ (1), we have $\xi^*(t)\pi^*(t) > \xi^c(t)\pi^c(t)$, $\forall t$. Hence $V^*(0) > V^c(0)$. Example 1 provides evidence for the last assertion of the Proposition. □

Proof of Proposition 3.

Substituting (15) into (1) and solving (24) at time $t = T$, we obtain the optimality condition $f_\ell(T) - w(T) = 0$, identical to that of the competitive firm. Consequently, $\overline{V}(T) = f(T) - w(T)\ell^D(T) > 0$ and $V_{ex}(T) = 0$. At time $t = T-1$, the first-order condition for (24) is

$$f_\ell(\ell(T-1),\varepsilon(T-1)) - w(T-1) \\ - \frac{u''(f(\ell(T-1),\varepsilon(T-1)))\,f_\ell(\ell(T-1),\varepsilon(T-1))}{u'(f(\ell(T-1),\varepsilon(T-1)))^2} \\ \times E[u'(f_\ell(\ell(T),\varepsilon(T)))\,\{f(T) - w(T)\ell^D(T)\}|\mathcal{F}_{T-1}] = 0\,.$$

Using the definition of V_{ex} and rearranging above yields (25) at time $T-1$. Continuing the backward induction, we obtain (25) for all $t = 1,\dots T-1$. The sufficient condition for concavity in footnote 11 guarantees that the labor solving (25) is the maximizer for (24). $\overline{V}(t)$, obtained on each step of the backward induction, is strictly positive because the firm's objective in (24) in strictly concave and $V(t) \geq 0$ ($V(t) < 0$ is ruled out since the firm can shut down at any time and thus increase its value to zero). Consequently, $V_{ex}(t) = E[\frac{\xi(t+1)}{\xi(t)}V(t+1)\,|\mathcal{F}_t]$ is strictly positive. This together with $A(t) > 0$ and $f_\ell(\ell,\varepsilon(t)) > 0$ implies that the right-hand side of (25) is strictly negative.
□

Lemma A.3.

Under the standard assumptions on preferences and production (see Sections 2.1, 2.2), and the regularity condition $-2\,\frac{u''(c)}{u'(c)} + \frac{u'''(c)}{u''(c)} < -\frac{f_{\ell\ell}(\ell,\varepsilon)}{f_\ell^2(\ell,\varepsilon)}$, $c = f(\ell,\varepsilon)$; $\forall\varepsilon$, ℓ, *(footnote 11), there exists a unique solution,* $\hat{\ell}(t) \in (0,\bar{\ell})$, *to (26).*

Proof of Lemma A.3. Since on each step of the backward induction, $\lim_{\ell\to 0} u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon) - v'(\bar{\ell}-\ell) + A(t)u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon)V_{ex} = \infty$ and $\lim_{\ell\to\bar{\ell}} u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon) - v'(\bar{\ell}-\ell) + A(t)u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon)V_{ex} = -\infty$ and since $u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon) - v'(\bar{\ell}-\ell) + A(t)u'(f(\ell,\varepsilon))f_\ell(\ell,\varepsilon)V_{ex}$ is decreasing in ℓ, there exists a unique solution, $\hat{\ell} \in (0,\bar{\ell})$, to (26). □

Proof of Proposition 4.

Since the right-hand side of (26) is strictly less than zero due to Proposition 3, and the right-hand side of (11) is zero, it follows from Lemma A.2 that $\hat{\ell}(t) > \ell^c(t)$. Consequently, $f(\ell,\varepsilon)$ and $w = v'(\bar{\ell}-\ell)/u'(f(\ell,\varepsilon))$ being increasing in ℓ, $\forall\varepsilon$, and the good market clearing yield the comparisons on f, c, w. The comparison on ξ follows from $\xi = u'(f(\ell,\varepsilon))$ being decreasing in ℓ, $\forall\varepsilon$.

To prove the remaining statements, define the equilibrium profit function $\Pi(\ell(t),\varepsilon(t)) \equiv f(\ell(t),\varepsilon(t)) - \frac{v'(\bar{\ell}-\ell(t))}{u'(f(\ell(t),\varepsilon(t)))}\,\ell(t)$. The first-order condition for maximization of $\Pi(\ell(t),\varepsilon(t))$ with respect to ℓ is satisfied by $\tilde{\ell}(t)$ such that

$$\begin{aligned} f_\ell(\tilde{\ell}(t),\varepsilon(t)) - &\frac{v'(\bar{\ell}-\tilde{\ell}(t))}{u'(f(\tilde{\ell}(t),\varepsilon(t)))} \\ &= -\frac{v''(\bar{\ell}-\tilde{\ell}(t))}{u'(f(\tilde{\ell}(t),\varepsilon(t)))} + \frac{A(t)\,f_\ell(\tilde{\ell}(t),\,\varepsilon(t))}{u'(f(\tilde{\ell}(t),\varepsilon(t)))}\,v'(\bar{\ell}-\tilde{\ell}(t)) \\ &> 0\,. \end{aligned} \tag{A.2}$$

Due to Lemma A.2, $\tilde{\ell}(t) < \ell^c(t)$, $\forall t, \varepsilon(t)$. It is straightforward to verify that for any ℓ such that $f_\ell - \frac{v'(\bar{\ell}-\ell)}{u'(f(\ell,\varepsilon))} \le 0$, $\forall\varepsilon$, $\Pi_\ell < 0$. This together with (26) implies that $\Pi_\ell(\ell^c(t)) < 0$ and $\Pi_\ell(\hat{\ell}(t)) < 0$ $\forall t, \varepsilon(t)$. Continuous function $\Pi_\ell(\cdot\,;\varepsilon)$ does not change sign on $[\ell^c(t),\hat{\ell}(t)]$, because if it did, there had to be a point $\breve{\ell}(t)$ on $[\ell^c(t),\,\hat{\ell}(t)]$ satisfying $\Pi_\ell(\breve{\ell}(t)) = 0$, which is not possible for we have shown that any such point has to lie to the left of $\ell^c(t)$. Consequently, $\Pi_\ell(\cdot\,;\varepsilon)$ is monotonically decreasing on $[\ell^c(t),\hat{\ell}(t)]$, and hence $\hat{\pi}(t) < \pi^c(t)$, $\forall t$. It then follows that since $\hat{\xi}(t) < \xi^c(t)$, $\hat{\xi}(t)\hat{\pi}(t) < \xi^c(t)\pi^c(t)$, and hence from (1) that $\hat{\xi}(t)\hat{V}(t) < \xi^c(t)V^c(t)$. This together with the argument behind the normalization adapted from the proof of Proposition 2, yields $\hat{V}(0) < V^c(0)$.

Since at time T the equilibrium conditions (11) and (26) of the competitive and the monopolistic time-consistent economies coincide, $\hat{\ell}(T) = \ell^c(T)$, yielding equalization of the remaining equilibrium quantities at time T. Finally, an example can be constructed to verify that $\hat{V}(t)$ can be lower or higher than $V^c(t)$. Example 2 of Section 3.2 demonstrates this under a modified set of assumptions on $v(h)$. □

Proof of Proposition 5.

Consider an arbitrary partition $t = 0, \Delta, 2\Delta, \dots, n\Delta = T$ of $[0,T]$. The consumer-worker's problem is now given by

$$\max_{c,\,\ell} E\left[\sum_{t=1}^{T}\Big(u(c(t)) + v(\bar{\ell} - \ell(t))\Big)\Delta\right]$$

$$\text{subject to}\quad E\left[\sum_{t=1}^{T}\xi(t)\Big(c(t) - w(t)\,\ell(t)\Big)\Delta\right] \le E\left[\sum_{t=1}^{T}\xi(s)\pi(s)\Delta\right],$$

where the flow of utility is defined over a rate of consumption and leisure. The above yields the first order conditions

$$\begin{aligned} u'(c(t))\Delta &= y\,\xi(t)\,\Delta\,, \\ v'(\bar{\ell} - \ell(t))\Delta &= y\,\xi(t)\,w(t)\,\Delta\,. \end{aligned} \tag{A.3}$$

For any Δ, the first-order conditions are equivalent to (4)–(5), consequently, the consumer demand facing the firm is the same as in the discrete case. The value of the firm is given by

$$V(t) = E\left[\sum_{s=t}^{T-\Delta}\frac{\xi(s)}{\xi(t)}\left(f(\ell(s),\varepsilon(s)) - w(s)\ell(s)\right)\Delta\,\Big|\,\mathcal{F}_t\right]. \tag{A.4}$$

The competitive firm is choosing ℓ^D to maximize $V(0)$ in (A.4) taking ξ as given; accordingly, its labor demand ℓ^D satisfies for all $t = \Delta, \dots, T-\Delta$

$$f_\ell(t)\,\Delta - w(t)\,\Delta = 0.$$

The pre-committed monopolist is choosing ℓ^D and ξ so as to maximize $V(0)$ in (A.4) subject to $\xi(t) = u'(f(\ell(t),\varepsilon(t)))/y$ (as implied by (A.3)). His labor demand ℓ^D satisfies

$$f_\ell(t)\,\Delta - w(t)\,\Delta = A(t)f_\ell(t)\pi(t)\Delta.$$

Since the equations above are independent of Δ, equations (34) and (35) obtain; in addition, the continuous-time limits of the discrete-time competitive and monopolistic pre-commitment equilibria are now given by the continuous-time analogs of (11)–(14) and (18)–(21), respectively, for all $t \in [0,T]$.

Similarly, the problem of the time-consistent monopolist is given by (24) with $V(t)$ specified in (A.4). The backward induction solution yields the following for the firm's labor demand ℓ^D

$$(f_\ell(t) - w(t))\,\Delta = -A(t)\,f_\ell(t)\,V_{ex}(t) \ \leq 0\,, \qquad t = \Delta, \dots, T-\Delta\,. \quad \text{(A.5)}$$

$\ell(t)$ is bounded from above by $\bar{\ell}$, and, anticipating equilibrium, since the right-hand of (A.5) is nonpositive, it is bounded from below by $\ell^c(t) > 0$ due to Lemma A.2. $\varepsilon(t)$ is bounded by assumption, hence $f(\ell,\varepsilon)$, $f_\ell(\ell,\varepsilon)$, $u'(f(\ell,\varepsilon))$ and $A(t)$ are bounded. As we take the limit as $\Delta \to 0$ in (A.5), given the boundedness, its left-hand side tends to zero, hence, so must the right-hand side. Given the boundedness of all quantities except $V_{ex}(t)$ on the right-hand side of (A.5), we must have $V_{ex}(t) \to 0$. Consequently, $\pi(t) \to 0$; in addition, given our boundedness argument, $V(t) \to 0$, and the firm's labor demand has to be such that $\ell^D(t)$ yields zero profit $\pi(t) = f(\ell^D(t), \varepsilon(t)) - w(t)\ell^D(t) = 0$, as is stated in (36). In the resulting equilibrium, $\hat{\pi}(t) = 0$ as well, hence (37). (37) yields the equilibrium labor, which in turn determines the remaining equilibrium quantities. □

Appendix B. Alternative Specifications of the Numeraire

Suppose that instead of the consumption good, we specified a basket of commodities consisting of α units of the consumption good and $(1-\alpha)$ units of labor as the numeraire, $\alpha \in (0,1)$. In other words, the price of this basket is normalized to 1:

$$1 = \alpha p(t) + (1-\alpha) w(t), \qquad \forall t = 1, \dots, T, \quad \text{(A.6)}$$

where $p(t)$, $w(t)$ are prices of the consumption good and labor, respectively, in units of the numeraire.

Under this numeraire, the price-taking consumer-worker's problem is given by

$$\max_{c,\,\ell} E\left[\sum_{t=1}^{T} u(c(t)) + v(\bar{\ell} - \ell(t))\right]$$

$$\text{subject to} \quad E\left[\sum_{t=1}^{T} \xi(t)\left(p(t)c(t) - w(t)\,\ell(t)\right)\right] \leq E\left[\sum_{t=1}^{T} \xi(t)\pi(t)\right],$$

where $\xi(t)$ in the state price density process in units of the numeraire basket. The first-order conditions of this problem are

$$u'(c(t)) = y\,\xi(t)p(t)\,,$$
$$v'(\bar{\ell} - \ell(t)) = y\,\xi(t)\,w(t)\,,$$

leading to

$$p(t)\frac{v'(\bar{\ell} - \ell(t))}{u'(c(t))} = w(t)\,. \quad \text{(A.7)}$$

The time-t value of the firm, in units of the numeraire, is given by

$$V(t) = E\left[\sum_{s=t}^{T} \frac{\xi(s)}{\xi(t)}\,\pi(s)\,\Big|\mathcal{F}_t\right], \tag{A.8}$$

where $\pi(s) = p(s)f(\ell^D(s), \varepsilon(s)) - w(s)\ell^D(s)$.

The competitive firm maximizes the time-0 value of the firm, taking all the prices as given. The optimal labor demand, for all $t = 1, \ldots, T$, of this firm is given by

$$\frac{1-(1-\alpha)w(t)}{\alpha} f_\ell(t) - w(t) = 0\,, \tag{A.9}$$

where we substituted the consumption good price from (A.6).

A monopolistic firm maximizes the quantity in (A.8) taking into account the consumer-worker's demand for its product

$$u'(f(\ell(t), \varepsilon(t))) = y\xi(t)p(t). \tag{A.10}$$

The pre-committed monopolist's problem of maximizing the time-0 value of the firm subject to (A.10), with $p(t)$ from (A.6), yields the optimal labor demand, for all $t = 1, \ldots, T$, solving

$$\frac{1-(1-\alpha)w(t)}{\alpha} f_\ell(t) - w(t) = A(t)\, f_\ell(t)\pi(t) \;>\; 0\,. \tag{A.11}$$

By backward induction, we find the time-consistent monopolist's labor demand, for all $t = 1, \ldots, T-1$, to be the solution to

$$\frac{1-(1-\alpha)w(t)}{\alpha} f_\ell(t) - w(t) = -A(t)\, f_\ell(t)V_{ex}(t) \;<\; 0\,, \tag{A.12}$$

where

$$V_{ex}(t) \equiv E\Big[\sum_{s=t+1}^{T} \frac{u'(s)}{u'(t)}\left[\frac{1-(1-\alpha)w(t)}{\alpha} f(s) - \frac{1-(1-\alpha)w(t)}{1-(1-\alpha)w(s)} w(s)\,\ell(s)\right]\Big|\mathcal{F}_t\Big] \quad \forall\, t = 1, \ldots, T-1\,.$$

At time T, the time-consistent monopoly optimal labor demand ℓ^D satisfies $\frac{1-(1-\alpha)}{\alpha} f_\ell(T) - w(T) = 0$. The only difference between expressions (A.9), (A.11), and (A.12) and the corresponding ones in the paper, (9), (17), and (25), respectively, is the presence of the additional term $\frac{1-(1-\alpha)w(t)}{\alpha}$ multiplying the marginal product of labor – this additional term is the price the consumption good. Since the right-hand sides of equations (A.11)–(A.12) are given by the same expressions as in the paper, all our partial equilibrium implications remain valid.

We now turn to equilibrium comparisons. First, from (A.7), (A.6) and the good market clearing, $c(t) = f(t)$, the equilibrium consumption good price and wage satisfy:

$$p(t) = \frac{u'(f(t))}{\alpha u'(f(t)) + (1-\alpha)v'(\bar{\ell} - \ell(t))} > 0\,,$$

$$w(t) = \frac{v'(\bar{\ell} - \ell(t))}{\alpha u'(f(t)) + (1-\alpha)v'(\bar{\ell} - \ell(t))} > 0\,. \tag{A.13}$$

To obtain the equation determining labor ℓ^c in the competitive equilibrium, we multiply (A.9) through by the positive quantity $\frac{\alpha}{1-(1-\alpha)w(t)}$ $\left(= \frac{1}{p(t)}\right)$ and then substitute in the equilibrium expression for the wage (A.13). Then, the equilibrium labor ℓ^c in the competitive economy solves

$$f_\ell(\ell^c(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \ell^c(t))}{u'(f(\ell^c(t), \varepsilon(t)))} = 0\,. \tag{A.14}$$

Analogously, multiplying (A.11) through by $\frac{\alpha}{1-(1-\alpha)}$ and using (A.13), we obtain the expression for the equilibrium labor ℓ^* in the monopolistic pre-commitment economy

$$\begin{aligned} f_\ell(\ell^*(t),\varepsilon(t)) &- \frac{v'(\bar{\ell} - \ell^*(t))}{u'(f(\ell^*(t), \varepsilon(t)))} \\ &= A(t) f_\ell(\ell^*(t), \varepsilon(t)) \left[f(\ell^*(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \ell^*(t))}{u'(f(\ell^*(t), \varepsilon(t)))}\, \ell^*(t) \right]. \end{aligned} \tag{A.15}$$

Finally, the equilibrium labor in the monopolistic time-consistent economy $\hat{\ell}$ is given by the solution to

$$\begin{aligned} f_\ell\,(\hat{\ell}(t), \varepsilon(t)) - \frac{v'(\bar{\ell} - \hat{\ell}(t))}{u'(f(\hat{\ell}(t), \varepsilon(t)))} &= -A(t)\, f_\ell(\hat{\ell}(t),\, \varepsilon(t)) \\ \times E\left[\sum_{s=t+1}^{T} \frac{u'(f(\hat{\ell}(s), \varepsilon(s)))}{u'(f(\hat{\ell}(t), \varepsilon(t)))} \left\{ f(\hat{\ell}(s), \varepsilon(s)) - \frac{v'(\bar{\ell} - \hat{\ell}(s))}{u'(f(\hat{\ell}(s), \varepsilon(s)))} \hat{\ell}(s) \right\} \Big| \mathcal{F}_t \right]. \end{aligned} \tag{A.16}$$

Comparison of (A.14), (A.15), and (A.16) to the corresponding equations determining the equilibrium labor in the paper (A.14), (A.15), and (A.16), respectively, reveals that equilibrium labor is invariant to the choice of the numeraire.

Once we have shown that the equilibrium labor is unchanged, it is straightforward to verify that the remaining results listed in Propositions 2 and 4 are robust as well, apart from the comparison of the state prices $\xi(\cdot)$. However, the more meaningful quantity to consider instead of $\xi(\cdot)$ in this case is the price of the consumption good relative to the numeraire basket. The prices of consumption in the monopolistic pre-commitment, competitive and time-consistent economies satisfy, for $t = 1, \ldots, T-1$,

$$p^*(t) > p^c(t) > \hat{p}(t)\,,$$

and $p^*(T) > p^c(T) = \hat{p}(T)$. Similarly to the textbook monopolist, the pre-committed monopolist in our model raises the price of the good it produces, while the time-consistent monopolist does the reverse.

Appendix C. The Case of a Monopolistic-Monopsonistic Firm

Let us suppose we allow the firm to also have power over the wage rate, i.e., to solve for all $t = 0, \dots T$:

$$\max_{\ell^D(s),\,\xi(s),\,w(s);\; s\geq t} V(t) \qquad \text{subject to } \xi(s) = u'\Big(f(\ell^D(s), \varepsilon(s))\Big)/y\,,$$
$$w(s) = \frac{v'(\bar{\ell} - \ell^D(s))}{u'(f(\ell^D(s),\, \varepsilon(s)))}\,, \quad \forall\, s = t, \dots, T.$$

The pre-committed monopolist-monopsonist (hereafter "monopsonist") only solves this problem at $t = 0$, while the time-consistent solves it backwards for $t = 0, \dots, T$. The first-order conditions for the pre-committed monopsonist are

$$f_\ell(t) - w(t) = A(t) f_\ell(t)\pi(t) + (A^\ell(t) + A(t) f_\ell(t))\ell(t)v'(t) \;> 0\,, \qquad \text{(A.17)}$$

and for the time-consistent monopsonist are

$$f_\ell(t) - w(t) = -A(t) f_\ell(t) V_{ex}(t) + (A^\ell(t) + A(t) f_\ell(t))\ell(t)v'(t)\,, \qquad \text{(A.18)}$$

where

$$A^\ell(t) \equiv -\frac{v''(t)}{v'(t)} \;> 0\,.$$

All other things (A, f_ℓ, π) being equal, (A.17) suggests that the wage effect acts in the same direction as the price effect, implying that the pre-committed monopsonist will decrease his labor (and output) even more than the monopolist, and in turn the value of the firm will increase more. For the time-consistent case, however, (A.18) suggests that the wage effect counteracts the price effect, implying again a lower labor input (and output) than the monopolist. If the price effect dominates, comparative static comparisons with the competitive case will be as in Proposition 4; if the wage effect dominates, they will be as in Proposition 2. The intuition for this extra term is quite clear; it is in the firm's interest to reduce wages, and to do so it will reduce its labor demand.

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Revealed Intertemporal Inefficiency

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Summary. In a stationary state, inefficiency is established if some marginal perturbation leads to a positive surplus for all commodities. We consider the case when, though the original perturbation itself results in positive and negative changes, an adequately chosen combination of such perturbations is conclusive. A characterization in terms of the positive roots of a polynomial is obtained.

1 Introduction

Let there be a differentiable model describing a system in a stationary state. In order to test the efficiency of that state, we introduce one or several marginal perturbations at some date, or on some consecutive dates, and contemplate the effects of this basic operation in terms of surplus. The paper examines when the operation allows us to conclude to inefficiency. Section 1 presents the result in the one-dimension case. Section 2 wonders if, in the presence of several commodities, inefficiency for each separate commodity is sufficient to conclude to global inefficiency. A general version of the theorem used in both cases is given in the mathematical appendix.

2 The One-Commodity Case

Considering the impact of marginal perturbations is the classical tool for studying inefficiency (Cass, 1972). In this section it is assumed that the basic operation affects one commodity. The implicit reason is not that there is one commodity only in the model, but that the marginal perturbation has been designed in order to have no influence on the surplus of the other commodities. In the next section, we retain another interpretation: there are positive or negative effects on the other commodities, but these effects are ignored, as if the supply of these commodities were illimited.

Starting from a stationary state, let the effects of the original perturbation be described by a variation $\varepsilon a = \varepsilon\,(a_0, ..., a_T)$ of the surpluses between dates t to $t + T$. Inefficiency is established if the operation leads to an increase of the surplus at every date, i.e. if the vector a is positive. The same if the vector is negative: it suffices to reverse the operation (linearity of marginal changes is used here). The interesting question occurs when, for instance, and up to a positive factor ε, the intertemporal changes in the surpluses are represented by the vector $a = (1, -2, 5)$ at the successive dates t, t+1 and t+2. Though the operation does not seem conclusive at first sight, let us consider the more complex transform which consists in repeating the same basic operation one period later and doubling its activity level ($y_0 = 1,\ y_1 = 2$). The overall result, represented in the last row of the table below, is semipositive: intertemporal inefficiency is revealed.

Table 1. An example of revealed inefficiency

dates	t	t+1	t+2	t+3
$y_0 = 1$	1	-2	5	
$y_1 = 2$		2	-4	10
overall surplus	1	0	1	10

Note that, since the changes ε(1, -2, 5) are linear approximations of the exact values, the zero component in the overall surplus is only a value close to zero and might be a small negative scalar. The choice $y_1 = 2.1$ would be more convincing, or the whole operation might be repeated with a lag, thus generating the positive surpluses (1, 0, 1, 10, 0) + (0, 1, 0, 1, 10). In general, the construction of a complex transform that generates a semipositive surplus is not obvious. However, what we need is not an expression of that complex transform itself, but a simple criterion that indicates the possibility or the impossibility to generate a semipositive surplus.

Let the basic operation, which remains unspecified and results in a sequence of marginal changes in the surpluses, be characterized by the activity level $y_0 = 1$. During the next periods, let the same operation be performed at levels $y_1, ..., y_N$ which may be positive or negative since they reflect changes in activity levels. We assume that the perturbations and the effects take place within a finite time. At date $t + k$, i.e. k periods after the initial perturbation, the change in the surplus amounts to

$$s_k = y_0 a_k + y_1 a_{k-1} + ... + y_k a_0 \tag{1}$$

where the coefficients which are not defined (viz., a_k for $k > T$ and y_k for $k > N$) are set equal to zero. Let us associate the formal polynomial $A(X) = \sum_{t=0}^{T} a_t X^t$ to the basic operation, the formal polynomial $Y(X) = \sum_{i=0}^{N} y_i X^i$

to the sequence of activity levels and, finally, the formal polynomial $S(X) = \sum s_t X^t$ to the sequence of surpluses. The algebraic relationship between these magnitudes is written $S(X) = A(X)Y(X)$. Therefore, inefficiency is revealed if $S(X)$ has semipositive coefficients (after multiplication by $1+X+...+X^M$, it might as well be required that its coefficients are positive). Let the technology be linear, but consider finite instead of infinitesimal changes. The difference is that the scalars y_t in formula (1) represent activity levels instead of changes in activity levels and, therefore, are semipositive (nonnegativity restrictions would also matter in the marginal case if the initial activity level is zero). Hence the following definition (to ease the reading, formal polynomials and formal series with semipositivity or positivity restrictions on the coefficients are denoted by letters P, Q or R):

Definition 1. *A marginal change described by the formal polynomial $\varepsilon A(X)$ reveals inefficiency in a finite time if and only if there exists a formal polynomial $Y(X)$ such that the formal equality*

$$A(X)Y(X) = P(X) \qquad (2)$$

holds, where P is semipositive. A non marginal change reveals inefficiency in a finite time if and only if there exist semipositive formal polynomials P and Q such that

$$A(X)Q(X) = P(X) \qquad (3)$$

It is immediately seen from equality (2) that a necessary condition is that the real polynomial $A(x)$ admits no positive root. Theorem 1 states that the condition is also sufficient. Curiously enough, condition (3) is not more restrictive than (2):

Theorem 1. *A marginal or non marginal change leading to the sequence of surpluses $(a_0, ..., a_T)$ reveals inefficiency in a finite time if and only if the polynomial $A(x) = \sum_{t=0}^{T} a_t x^t$ has no positive root.*

Proof. Let $a_0 > 0$ and apply Theorem 4 of the mathematical appendix to the case when the matrices A_t have dimension one and are the coefficients of the polynomial $A(x)$. The alternative between conditions (i) and (iii) of Theorem 4 means that either $A(x)$ admits a positive root or $A(x)$ is the ratio between two polynomials with semipositive coefficients, as in (3).

Up to now we have proceeded to a finite number of successive changes. A justification of the hypothesis is that, if there were infinitely many nonzero values of y_t, y_t might tend to infinity and the marginal approximation would hold no longer. However, the objection is avoided if we restrict the attention to uniformly bounded sequences $\{y\}$. For instance, the perturbation leading to the surpluses $\varepsilon(a_0,\ a_1 = -1)$ with $a_0 \geqslant 1$ cannot reveal inefficiency in a finite time, but it does so in an infinite time, by repeating the operation for all dates at the same activity levels. The example suggests that, in infinite horizon, what matters is the position of the real roots of $A(x)$ with regard to 1. The following notations are adopted: bars refer to formal series, as opposed to polynomials, and $\overline{U}$ means that the coefficients are uniformly bounded.

Definition 2. *A marginal change reveals inefficiency in a finite or infinite time if and only if the formal equality*

$$A(X)\overline{U}(X) = \overline{P}(X) \tag{4}$$

holds, where the formal series $\overline{U}$ has uniformly bounded coefficients and the formal series $\overline{P}$ is semipositive.

Theorem 2. *A marginal change reveals inefficiency if and only if its associated polynomial $A(x)$ has no root in $]0,1[$ and the root $x = 1$, if any, is simple.*

Proof. If $A(x)$ has no root in $]0,1[$ and, possibly, a simple root at $x = 1$, $A(X)$ is written

$$A(X) = B(X)\,\Pi_i\left(1 - \rho_i^{-1}X\right)[1 - X] \tag{5}$$

where $B(x)$ has no positive root, the ρ_is are the roots of A greater than 1 $\left(\text{let } 1 < \rho < \rho_1 \leq \cdots \leq \rho_k\right)$ and the presence of the factor $[1 - X]$ is optional. According to Definition 1 and Theorem 1, an equality

$$B(X)\,Q_1(X) = P_1(X) \tag{6}$$

holds, where $P_1(X)$ and $Q_1(X)$ are semipositive polynomials. The formal inverse $\sum \overline{p}_t X^t$ of the polynomial $\Pi_i\left(1 - \rho_i^{-1}X\right)$ is positive and written

$$\begin{aligned}\Pi_i\left(1 - \rho_i^{-1}X\right)^{-1} &= \Pi_i\Big(\sum_t \rho_i^{-t}X^t\Big)\\ &\leq \Pi_i\Big(\sum_t \rho_1^{-t}X^t\Big) = \Pi_i(1 - \rho_1^{-1}X)^{-1} = (1 - \rho_1^{-1}X)^{-k}\end{aligned}$$

Therefore

$$\overline{p}_t \leq \frac{(t + k - 1)!}{t!\,(k - 1)!}\rho_1^{-t} = v_t$$

Since $v_{t+1}/v_t = \rho_1^{-1}(t + k)\,(t + 1) < \rho^{-1}$ for t great enough, there exists some positive scalar m such that $\overline{p}_t \leq v_t < m\rho^{-t}$ for any t. As a consequence the series $\Pi_i\left(1 - \rho_i^{-1}X\right)^{-1}$ or, if the case occurs,

$$\overline{U}_1(X) = \Pi_i\left(1 - \rho_i^{-1}X\right)^{-1}[1 - X]^{-1} \tag{7}$$

has positive and uniformly bounded coefficients. By multiplying equalities (5), (6) and (7) side by side, equality

$$A(X)\left[Q_1(X)\,\overline{U}_1(X)\right] = P_1(X)$$

is obtained and condition (4) is met. Hence, inefficiency is revealed.

Conversely, let equality (4) hold. Assume first that A admits a root x_0 in $]0,1[$. The real series $\overline{U}(x)$ converges at $x = x_0$ and a contradiction is obtained between equality $\overline{P}(x_0) = A(x_0)\overline{U}(x_0) = 0$ and the semipositivity of $\overline{P}$. Assume now that A admits no root x_0 in $]0,1[$ but that 1 is a root of order k $(k \geq 2)$. Then

$$A(X) = (1-X)^k B(X) \Pi_i \left(1-\rho_i^{-1}X\right)$$

where $B(x)$ has no positive root and $\rho_i > 1$. There exist positive polynomials P_2 and Q_2 such that $P_2 = BQ_2$. Replace $A(X)$ by the above expression in (4) and multiply both members of (4) by the positive series

$$\overline{R}(X) = (1-X)^{2-k} Q_2(X) \Pi_i \left(1-\rho_i^{-1}X\right)^{-1}$$

Equality

$$(1-X)^2\overline{U}_2(X) = \overline{P}_2(X)$$

is obtained, where $\overline{U}_2 = P_2\overline{U}$ has uniformly bounded and the coefficients of $\overline{P}_2 = \overline{P}\,\overline{R}$ are positive. (In other terms, a comparison with (4) shows that the original problem has been reduced to the specific case $A(X) = (1-X)^2$.) Let $\overline{S}(X) = (1-X)\overline{U}_2(X)$. The inequality $(1-X)\overline{S}(X) > 0$ implies $0 < \overline{s}_0 < \overline{s}_1 < \overline{s}_2 < \cdots$. Therefore the tth coefficient of $\overline{U}_2(X) = (1-X)^{-1}\overline{S}(X) = (\sum_t X^t)\overline{S}(X)$ is greater than $t\overline{s}_0$, and a contradiction is obtained with the uniform boundedness hypothesis.

In the case of finite but bounded changes, instead of marginal changes, the series $\overline{U}$ in Definition 2 must moreover be semipositive. An inspection of the above proof shows that the characterization given in Theorem 2 is unchanged.

3 Several Commodities

We now consider the case when the initial perturbation affects several commodities, for instance two commodities A and B during several periods. The effects are described by the simultaneous changes $\varepsilon(a_0,\ a_1, ...,\ a_T)$ for commodity A and $\varepsilon(b_0,\ b_1, ...,\ b_T)$ for B. The question of detecting global inefficiency is that of giving the vectoral version of Theorems 1 and 2 and, therefore, the expected tool is a theorem on vectors, as the one stated in the appendix. We explore an alternative path and consider the question component by component.

An initial condition is required. Assume first that a_0 and b_0 are both nonzero, i.e. the first real effects of the changes occur at the same date. If a_0 and b_0 have opposite signs, this will also be the case for any complex transform derived from the basic operation. Then the surpluses in both commodities cannot be simultaneously positive and it is not possible to conclude to inefficiency. In the more complex case when the initial effects on A and B are not simultaneous, say $a_0 > 0$ but $b_0 = b_1 = 0 \neq b_2$, the initial condition is transformed as follows: the first activity levels $(y_0 = 1, y_1, y_2)$ must be such that

$$y_0 a_0 > 0 \qquad y_0 a_1 + y_1 a_0 \geqslant 0 \qquad y_0 a_2 + y_1 a_1 + y_2 a_0 \geqslant 0 \qquad y_2 b_2 \geqslant 0$$

These inequalities require minimum levels for y_1 and y_2, but the sign of the minimum level of y_2 is undetermined.

To avoid these discussions, we return to the simplest case and assume simultaneous initial changes: $a_0 > 0$ and $b_0 > 0$. Apart from that initial condition, let us consider the surpluses $(a_0, ..., a_T)$ and $(b_0, ..., b_T)$ separately and assume that, when each good is considered in isolation, some marginal perturbation reveals inefficiency in a finite time (the result for infinite time would be similar). Can we conclude that the change reveals inefficiency for both commodities considered simultaneously? Normally not: the activity levels revealing inefficiency for A differ from those revealing inefficiency for B, whereas the same activity levels should apply to both commodities.

In formal terms, the inefficiency relative to A is written

$$\exists\, Y_1(X) \qquad A(X)Y_1(X) = S_1(X) \qquad\qquad \text{with } S_1 > 0 \tag{8}$$

and the one relative to B is

$$\exists\, Y_2(X) \qquad B(X)Y_2(X) = S_2(X) \qquad\qquad \text{with } S_2 > 0 \tag{9}$$

Global inefficiency, with A and B considered as joint products, is written

$$\exists\, Y(X) \qquad A(X)Y(X) > 0 \qquad \text{and} \qquad B(X)Y(X) > 0 \tag{10}$$

There is no obvious way to derive (10) from (8) and (9). A curiosum is that the implication does hold:

Theorem 3. *If $a_0 b_0 > 0$, the revelation of inefficiency for commodities A and B separately suffices to reveal inefficiency for the system as a whole.*

Proof. If the inefficiency of A is revealed, the equivalence between conditions (2) and (3) shows that it is possible to write down an equality of the type (8) holds where, moreover, Y_1 is semipositive (if $a_0 > 0$). The same for commodity B (if $b_0 > 0$). Then condition (10) is met by $Y(X) = Y_1(X)Y_2(X)$.

4 Conclusion

A marginal perturbation in a stationary system may result in positive and negative variations of the surplus over several periods. It reveals inefficiency if an adequately chosen sequence of such perturbations results in positive variations only. Though the adequate combination may be difficult to find, its existence has been characterized in terms of the roots of a polynomial associated with the perturbation. In the presence of a multidimensional perturbation, an initial condition and inefficiency for each commodity separately suffice to establish global inefficiency.

5 Mathematical Appendix

Theorem 4 is a slightly adapted version of a result established by Bidard (1999) to study the theory of fixed capital. A sequence of nonnegative vectors is called 'essential' when T consecutive vectors are not equal to zero. The hypothesis

$$\nexists\ y_0 > 0 \qquad -A_T y_0 \geqslant 0 \tag{11}$$

implies that any nonzero sequence of nonnegative vectors satisfying property (ii) below is essential. Vectors with semipositivity or positivity restrictions are denoted by letters p, q, r or v.

Theorem 4. *Let $A_0, A_1, ..., A_T$ be square matrices with A_0 semipositive and indecomposable and A_T satisfying (11). The following properties (i) and (ii) are equivalent. Moreover, either (i) or (ii) holds, or property (iii) holds:*

(i) There exists a positive scalar λ and a semipositive vector v such that inequality (12) holds:

$$\exists\ \lambda > 0 \qquad \exists\ v > 0 \qquad -\left(\sum_{t=0}^{T} \lambda^t A_t\right) v \geqslant 0 \tag{12}$$

(ii) There exists an essential sequence of nonnegative vectors v_t such that the formal product

$$\sum r_t X^t = -\left(\sum_{t=0}^{T} A_t X^t\right)\left(\sum v_t X^t\right) \tag{13}$$

has nonnegative elements for any $t \geqslant T$.

(iii) There exist semipositive row-vectors $(\widetilde{q}_0, ..., \widetilde{q}_N)$ such that the formal product

$$\sum_{j=0}^{N+T} \tilde{p}_j X^j = \left(\sum_{i=0}^{N} \widetilde{q}_i X^i\right)\left(\sum_{t=0}^{T} A_t X^t\right). \tag{14}$$

is positive.

Proof. If property (i) holds, then property (ii) holds for the choice $v_t = \lambda^{-t} v$. Conversely, assume that property (ii) holds. Consider the set H made of the 'heads' $\{(v_0, v_1, ..., v_T)\} \subset R_+^{nT}$ of all vector sequences satisfying condition (13), with $r_t \geq 0$ for $t \geq T$. H is a convex cone. Let K be the normalized heads, i.e. the intersection of H with the unit simplex. Note the following uniform boundedness (UB) property: Because of the assumptions on A_0, inequality $-(A_0 v_t + A_1 v_{t-1} + ... + A_T v_{t-T}) \geqslant 0$ requires that $\|v_t\|$ is not too great with regard to $\|v_{t-1}\|, ..., \|v_{t-T}\|$; therefore, by induction on t, $\|v_t\|$ is uniformly bounded for all sequences whose head belongs to K. A consequence of (UB) is that K is closed.

Consider the translation correspondence γ defined on H by $\gamma(v_0, ..., v_T) = \{v_{T+1}; (v_1, ..., v_T,\ v_{T+1}) \in H\}$. It has compact convex values. Let the correspondence $\delta : K \longrightarrow K$ be defined by $\delta(v_0, ..., v_T) = \{\lambda(v_1, ..., v_T,\ \gamma(v_0, ..., v_T))\}$, where λ is a normalization factor (λ is well-defined because $(v_0, ..., v_T) \in K$ excludes $v_1 = ... = v_T = 0$, thanks to assumption (11)). δ meets the assumptions of the Kakutani theorem and, therefore, admits a fixed point

$$\exists\ (v_0, ..., v_T) \in H \qquad \exists \lambda > 0 \qquad v = v_0 = \lambda v_1 = ... = \lambda^t v_T \tag{15}$$

By definition of K, the Tth term of the formal sequence $-\left(\sum A_t X^t\right)\left(\sum v_t X^t\right)$ is semipositive:

$$-(A_T v_0 + A_{T-1} v_1 + ... + A_0 v_T) \geqslant 0. \tag{16}$$

Property (i) results from (15) and (16), therefore the equivalence of (i) and (ii) is proved.

Assume now that properties (ii) and (iii) hold simultaneously, with $v_m > 0$. Equalities (13) and (14) lead to a contradiction concerning the sign of the coefficient of X^{t+N+m} in the product $\left(\sum \widetilde{q}_i X_i\right)\left(\sum A_t X^t\right)\left(\sum v_t X^t\right)$.

Finally, assume that property (ii) does not hold. Suppose that, for any N, there exists a finite sequence $\left(v_0^{(N)}, ...,\ v_t^{(N)}, ...\right) \geqslant 0$, with $\|v_0\| = 1$, such that the vectors r_t defined by (13) are semipositive for any t in the interval $[T, T+N]$. By the (UB) property, every $\|v_t\|$ is bounded independently of N. When N tends to infinity, let us extract successive converging subsequences and construct an infinite sequence $(\overline{v}_0, ..., \overline{v}_t, ...)$ as a Cantor diagonal: $\overline{v}_0 = \lim_{N \in N_0} v_0^{(N)}$, $\overline{v}_1 = \lim v_1^{(N)}$, etc. This limit sequence satisfies property (ii). This being excluded by hypothesis, we conclude that there exists some N such that the system of linear inequalities $r_T \geqslant 0, ..., r_{T+N} \geqslant 0$ has no semipositive solution. In matricial terms

$$\nexists v > 0 \qquad Mv = -\begin{bmatrix} A_T & \cdots & A_0 & & & 0 \\ & A_T & & A_0 & & \\ & & \ddots & & \ddots & \\ 0 & & & A_T & & A_0 \end{bmatrix} v \geqslant 0$$

According to the Theorem of the Alternative, there exists a semipositive row-vector $\widetilde{q}$ such that $\widetilde{q}M << 0$. Therefore, property (iii) holds.

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Volatility and Job Creation in the Knowledge Economy

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Summary. Sectors with increasing returns to scale have been shown to amplify business cycles exhibiting more volatility than others [13]. Our hypothesis is that this volatility could be a cause of the "jobless recovery" suggesting policies for employment generation. To test this hypothesis we introduce a general equilibrium model with involuntary unemployment. The economy has two sectors: one with increasing returns that are external to the firm and endogenously determined – the knowledge sector – and the other with constant returns to scale. We define a measure of employment volatility, a 'labor beta' that is a relative of the 'beta' used in finance. A 'resolving' equation is derived from which it is proved that increasing return sectors exhibit more employment volatility then other sectors. The theoretical results are validated on US macro economic data of employment by industry (2-3 digits SIC codes) of the 1947-2001 period, showing that the highest 'labor betas' are in the service sectors with increasing returns to scale. Policy conclusions are provided to solve the puzzle of the 'jobless recovery', where small firms in the services industry play a key role. We conclude with policy recommendations on how to create jobs in the knowledge economy.

1 Introduction

In a recent article [13] we showed that increasing returns to scale sectors amplify the business cycle: they grow faster than others during expansions, and contract faster during downturns. In this sense they exhibit more 'volatility' than other sectors[1]. In this paper, we extend the earlier model to analyze

[1] The effect of the business cycle on IRS sectors has not been analyzed in the literature. Real Business Cycle models include IRS in order to generate cyclical productivity (see for example S. Basu and J. Fernald [3] and [4] for a review of the literature). In international trade, IRS is seen as determining the pattern and the factor content of trade (Krugman [23], [24], [25], [26], Panagariya [27], Antweiler et al. [1]). In growth literature, learning by doing leads to endogenous growth in the economy (see Arrow[2], Romer[30], and Rivera-Batiz and Romer[29]). In theoretical macrodynamic general equilibrium models, IRS may lead to unstable

volatility of employment. Our hypothesis is that this volatility could be a cause of the "jobless recovery"[2] suggesting policies for employment generation.

Chichilnisky-Gorbachev (2004) measured volatility using a new concept, a 'real beta', which is a statistical relative of the 'beta' used in financial markets, and measures real macro data on output rather than in stock prices. We used a general equilibrium model with two sectors, one with constant returns to scale (CRS) and the other with *external* increasing returns to scale (IRS), similar to that in Chichilnisky [9], [10] and [11]. In both sectors the firms are competitive, but the 'knowledge sector'[3] has increasing returns to scale that are *external* to the firm and are endogenously determined. The externalities arise from "learning by doing," analogous to Arrow (1962), and from the free transfer of skills from one firm to the other through job turnover or freely shared skills or R&D. Increasing returns are endogenous because they depend on the output in the knowledge sector in equilibrium[4]. We showed that IRS sectors exhibit more volatility than other sectors, an outcome we tested empirically using US data[5]; as these sectors become a larger share of the economy their volatility increases economic uncertainty. These results are related to the 'virtuous and vicious' cycles in economies with increasing returns to scale that were developed earlier in Chichilnisky and Heal [14] and in Heal [20].

The model in Chichilnisky-Gorachev (2004) had *full employment* in equilibrium. Extending this model, we introduce here a general equilibrium model with *involuntary unemployment* and two sectors, one with external IRS and

systems and may generate "vicious" and "virtuous" cycles in the economy (see Chichilnisky and Heal [14], and Heal [20], [19]).

[2] This is the second jobless recovery recorded in the US, the first was in 1990-1991.

[3] To identify "knowledge sectors" we used benchmarks from the World Knowledge Competitiveness Index introduced by Robert Huggins Associates, London, UK. The World Knowledge Competitiveness Index 2004 is an integrated and overall benchmark of the knowledge capacity, computed utilizing 19 knowledge economy benchmarks, including employment levels in the knowledge economy, patent registrations, R&D investment by the private and public sector, education expenditure, information and communication technology infrastructure, and access to private equity (www.hugginsassociates.com).

[4] Chipman (1970)[16] provides an excellent general equilibrium model with external economies of scale. The main difference with our model is in that his has only one factor of production and economies of scale are *exogenously* given, whereas here (and in Chichilnisky 1993, 1994, and 1998) there are two factors and the extent of IRS is endogenously determined in one sector of the economy– the other has constant returns.

[5] Our hypothesis is difficult to prove in complete generality, because the data of the last ten years is complex: industry classifications have changed and, as is well know, when using nominal data the increasing returns sectors (whose prices drop when production expands) are underrepresented in GDP. This observation is related to comments by William Baumol on productive and unproductive sectors, and to the November 2002 publication of Survey of Current Business, U.S. [35].

the other with CRS. The external IRS sectors or knowledge sectors are often characterized by having many small competitive firms. According to the Small Business Administration Office of Advocacy, small firms, defined as having 500 employees or less, represent 50 percent of all employment but generate about 60-80 percent of net jobs in the US [40]. Therefore, while IRS sectors are more volatile, small firms within IRS sectors create more jobs than they destroy and more jobs than larger firms, as pointed out by Chichilnisky (2004).

To achieve a meaningful representation of involuntary unemployment in a general equilibrium context, in the model presented here we postulate that labor supply is 'produced' as an increasing function of several variables rather than a constant endowment of time (24 hours) possessed by each individual from birth. The amount of labor supplied is constrained by the amount of physical energy, health, free time, skills, effort, and education a person possesses, which in turn relate to wages. An implication of this is that in a period of structural change, as labor skills become outdated, involuntary unemployment increases, an observation that has been made by the Federal Reserve Bank of New York in 2003 [18]. The formulation of our model of 'produced labor' supply has the same mathematical structure as the well-known Shapiro and Stiglitz efficiency wage model [32][6]; therefore, we do not provide a separate derivation of the labor market here, but use the Shapiro-Stiglitz model for this purpose as presented in the Appendix.

The article proceeds as follows: first we provide a general equilibrium model with involuntary unemployment and solve it analytically finding all prices, the level of unemployment, and production levels in equilibrium by means of a single 'resolving' equation; second we define the 'labor beta' measure of employment volatility for the different sectors of the economy, and prove formally that employment in external IRS sectors is more volatile than in other sectors; third, we provide details of the data used and validate the results on the observed 'labor betas' during the 1948-2001 periods; fourth, we discuss the jobless recovery in the context of this model. We show that small firms and the service sector play a crucial role and conclude with policy recommendations on how to create jobs in the knowledge economy.

2 General Equilibrium Model with External Economies of Scale and Involuntary Unemployment

Internal and External Increasing Returns to Scale

A firm or an industry has *increasing returns to scale* (IRS) when unit costs fall with increases in production[7]. Economies of scale are *internal* to the firm when

[6] The produced labor equation that we use here, see section 2.03, is equivalent to the Non-shirking condition of Shapiro-Stiglitz, developed formally in the context of our model in the Appendix.

[7] Sometimes they are defined by 'average' unit costs that decrease with production.

a firm becomes more productive as its own size increases, i.e. more efficient, in utilizing its resources as is typical to firms with large fixed costs such as aerospace, airlines, and oil refineries[8]. In contrast, increasing returns to scale are *external* to the firm when the increased productivity comes about as a result of decreasing unit costs at the level of the industry as a whole. In the latter case, each firm could have constant unit costs as its production increases, and behave competitively[9]. Yet as the industry as a whole expands, positive *externalities* among the firms are created leading to increased productivity for all firms in the industry[10]. The free movement of skilled workers from one firm to another can have this effect, as a firm may benefit at no cost from training a worker received in another firm[11]. Equally, a firm can benefit from unspecific research and development innovations developed in other firms, which are accessible to it at little or no cost. These positive 'knowledge spillovers' often originate from innovations generated during the course of production. As the new knowledge spreads to all the firms in the industry, total productivity in the industry increases and unit costs fall[12].

General Equilibrium with External Increasing Returns to Scale

The economy produces and trades two goods B and I. B is a traditional constant returns to scale (CRS) industry, whereas I is produced under *external* increasing returns to scale (IRS). Both goods are produced using two inputs, labor L and capital K, and the firms in each industry are perfectly competitive. They minimize their costs given the market prices. Firms production functions are given as:

[8] This type of increasing returns can lead to monopolistic competition due to high entry costs.

[9] *External* IRS is consistent with small competitive firms; *internal* IRS depends instead on large size of firms which is typically inconsistent with competitive behavior.

[10] For example, in the period between 1990 and 2000, the expansion in output in the computer hardware industry led to yearly doubling of the computing power available per dollar, leading to an exponential increase in CPUs per dollar (a standardized measure of processing power) and to the corresponding rapid increase in demand and consumption of CPUs across the entire economy.

[11] Workers in the knowledge sectors move between firms more than others, on average every two years or less.

[12] Any industry that depends on knowledge or skilled labor could benefit from such knowledge spillovers and external economies of scale. In the growth literature related phenomena are known as 'learning by doing', a concept introduced by Arrow [2] and developed further by Romer [30] in a one sector growth model. 'Learning by doing' often refers to increasing returns that are internal to the firm. We focus instead on increasing returns that are external to the firm, endogenously determined and internal to one industry within a general equilbirum model with two goods and two factors, and in which a second sector has constant returns to scale.

$$B^s = L_1^{\alpha} K_1^{1-\alpha} \text{ and } I^s = \gamma L_2^{\beta} K_2^{1-\beta} \tag{1}$$

where $\alpha, \beta \in (0,1)$. L_1, K_1 are inputs in the B sector, and L_2, K_2 are inputs in the I sector. The total amount of labor and capital in the economy are N and K^s respectively. The parameter γ in the production function for I is taken as a constant by each firm within this industry. However, at the industry level, γ is endogenously determined and increases with the output of I, i.e. $\gamma = \gamma(I^s)$. For example, $\gamma = I^{\sigma}$, $0 < \sigma < 1$. In this case, when externalities are taken into account the production function for this sector is $I^s = L_2^{\frac{\beta}{1-\sigma}} K_2^{\frac{1-\beta}{1-\sigma}}$, although at the firm's level the technology that determines the firm's behavior is $I^s = \gamma L_2^{\beta} K_2^{1-\beta}$. Observe that $\sigma > 1$ leads to negative marginal products, while $\sigma < 0$ leads to decreasing returns. Therefore, when the sector I has increasing returns, σ satisfies $0 < \sigma < 1$, which we now assume[13]. Notice that returns to scale in the sector I are endogenous because the parameter $\gamma = I^{\sigma}$ is unknown until the equilibrium value of I is determined.

Prices for I and B are p_I and p_B, respectively, and I is a numeraire good, or $p_I = 1$, then

$$Y^s = I^s + p_B B^s \tag{2}$$

By assumption each firm has *external* IRS so it forecasts a scale parameter γ as exogenously given, and solves the cost minimization problem with first order conditions as:

$$w = p_B \alpha (L_1)^{\alpha-1} (K_1)^{1-\alpha} \text{ and } r = p_B (1-\alpha)(L_1)^{\alpha}(K_1)^{-\alpha} \text{ in sector } B \tag{3}$$

$$w = \gamma \beta (L_2)^{\beta-1} (K_2)^{1-\beta} \text{ and } r = \gamma (1-\beta)(L_2)^{\beta}(K_2)^{-\beta} \text{ in sector } I$$

We assume that both capital and labor markets are perfectly competitive, thus the factor prices paid in B sector must equal that of I sector. The γ's will be determined by equilibrium output and the γ's that the firms projected will be the ones obtained in general equilibrium[14].

'Produced' Labor and Involuntary Unemployment in General Equilibrium

Our next task is to achieve a representation of involuntary unemployment in a general equilibrium context. To this end we postulate a 'produced' labor supply curve that is an increasing function of wages (see Figure 2 below). It is not the purpose of this article to provide an explicit derivation of the 'produced labor' supply curve; instead, we use as a model for this 'produced

[13] From equation (19) the marginal product of labor is $MPL = \frac{1}{1-\sigma}(\Phi(p_B)^{\frac{\sigma}{1-\sigma}})(\frac{\partial \Phi}{\partial \bullet_B})(\frac{\partial \bullet_B}{\partial L})$. When $\sigma > 1$ marginal product is negative. When σ is less than or equal to 0, the I sector exhibits decreasing or constant returns to scale, and $0 < \sigma < 1$ is IRS.

[14] This notion is similar to that of rational expectations assumptions for technology and production.

labor' supply the non-shirking condition that arises from the efficiency wage theory of Shapiro and Stiglitz [32] (see Figure 1), the connection between the two is developed below and in the Appendix, see Figure 1.

The 'produced' labor supply function is taken as exogenously given by workers (hence it is 'involuntary'). It represents not their willingness to work but rather the ability to do a certain job at a certain wage given for example the current skill or education level. In the Appendix, we consider a specific and familiar representation of this 'produced labor' function which delivers involuntary unemployment as in Shapiro-Stiglitz' efficiency wages article [32]. In that paper, the non-shirking condition (NSC) as illustrated in Figure 1 and derived in the Appendix[15] acts as the equivalent to our 'produced' labor supply function. Shapiro-Stiglitz observe that in the environment characterized by imperfect information (in their case firms are unable to tell the actual effort level of their workers), firms are forced to pay a higher than market clearing wage in order to keep workers from shirking, hence the non-shirking condition and the resulting involuntary unemployment.

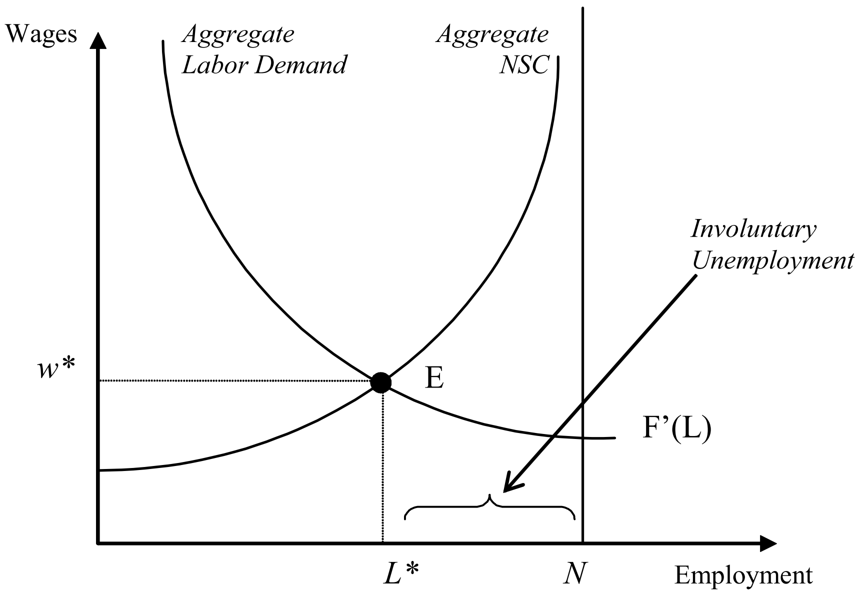


Fig. 1. Labor Market Equilibrium in General Equilibrium Model with Involuntary Unemployment Arising from Asymmetric Information from Shapiro and Stiglitz (1984), see Appendix

[15] This figure is reproduced from Figure 2 of Shapiro-Stiglitz paper [32].

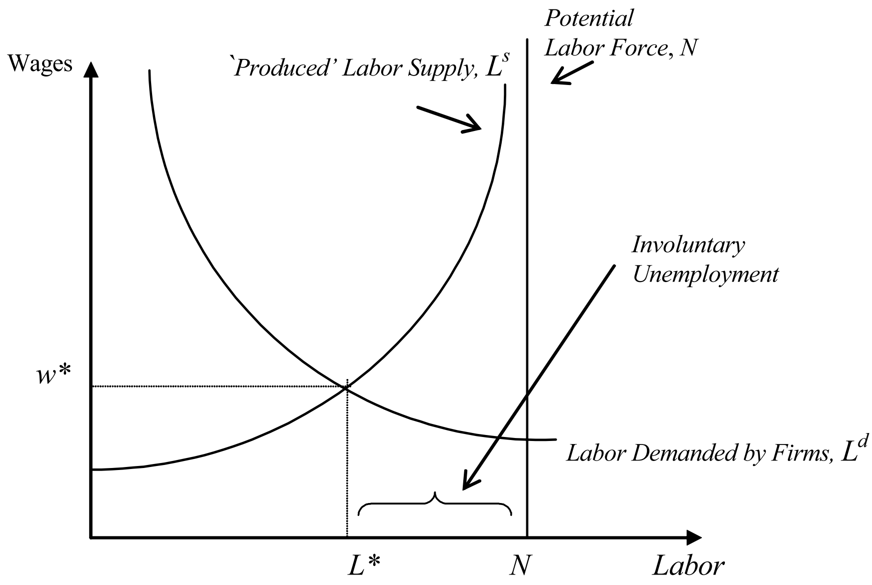


Fig. 2. Labor Market Equilibrium in General Equilibrium Model with Involuntary Unemployment Arising from 'Produced Labor' (analogous to Shapiro and Stiglitz model of 1984, see Figure 1)

The 'produced labor' supply function increases with wages because workers are able to supply more labor at higher wages since acquisition of skills requires resources. For example, in the efficiency wage models, there is minimal threshold that workers must reach in order to be considered productive labor units (due for example to minimal income required to sustain persons' minimal health). As already mentioned, we do not derive explicitly this function, taking instead Shapiro-Stiglitz formulation as a foundation.

The concept of involuntary unemployment is illustrated in Figure 2[16]. The vertical line represents total potential labor force in the economy or the total number of individuals, N. Potential labor force, N, is different from the 'produced' labor supply, which is a positively sloped curve, L^s, because as mentioned above it is an exogenously give function that depends on the current levels of workers' skills or education. L^s is a function of wages, w, and prices, $p_{B,}$ or $L^s = L^s(w, p_B)$. Labor demand, L^d, is a decreasing function of wage and is derived from firms' cost minimization problem or after solving for L in equation (3).

In equilibrium, due to the shape of labor supply curve, there will exist involuntary unemployment, represented by the difference in labor employed L^*

[16] Notice that Figure 2 is a special case of Figure 1.

and potential labor supply, N. Unemployment, $(N - L^*)$, is involuntary since there are workers who are willing to work at a lower wage but because they cannot credibly promise to do the job since they do not have the required-skills. Thus, they will not be hired. According to Shapiro and Stiglitz' model, unemployment is involuntary because workers who are willing to work at a lower than equilibrium wage cannot make a credible promise to exert effort or not to shirk.

Resolving the General Equilibrium Model

This section draws on the results of Chichilnisky [9] and Chichilnisky-Gorbachev [13] to solve our general equilibrium model with involuntary unemployment. To solve this model, there are four prices to be determined: the goods prices p_B^* and p_I^*, and the two factor prices, wages and return on capital, w^* and r^*. The quantities to be determined in an equilibrium are: the use of factors in each sector $K_{1,}^* K_{2,}^* L_{1,}^* L_2^*$, the outputs of the two goods B^{s*} and I^{s*}, the parameter γ^* determining the external economies of scale in I, and the demand for goods B and I, B^{d*} and I^{d*}. Thus the model has 13 endogenous variables to be computed, p_B^*, p_I^*,w^*, $r^*, K_{1,}^* K_{2,}^* L_{1,}^* L_2^*, B^{s*}, I^{s*}, \gamma^*, B^{d*}$ and I^{d*}.

We solve the model by finding an explicit function of a single variable, p_B, (called a 'resolving' equation) and 5 exogenously given parameters of the economy: α, β, σ, N and K^s. To obtain the resolving equation we write the market clearing condition in the I market, demand equals supply, and find a way to express it as a function of only one variable: p_B. Solving this equation gives the equilibrium value of p_B^*,from which all other endogenous variables listed above can be found[17]. Since the model has constant returns to scale at the level of the firms[18], we derive the equilibrium relations between supplies and prices from the marginal conditions, clearing of labor market and full employment of capital.

For simplicity[19], demand functions for B given as a function of initial endowments and prices:

$$B^d = B^d(N, K^s, p_B) \tag{4}$$

In equilibrium all markets clear, or:

$$B^{s*} = B^{d*} + X_B^* \text{ } (B \text{ market clears}) \tag{5}$$

[17] Notice that we have not defined supply behavior outside of an equilibrium; in particular, there is no information for carrying out stability analysis.

[18] As mentioned in the text, individual firms in both sectors of the economy exhibit CRS. Firms that belong to external IRS sector don't know that positive externalities are generated by their production or hiring decisions. Thus, firms in external IRS sectors, take γ of (1) equation as given.

[19] The results of the model do not depend on the demand specification. But in the Appendix, we solve for a specific demand function.

$$I^{s*} = I^{d*} + X_I^* \text{ (} I \text{ market clears)}$$

where X_B^* and X_I^* are equilibrium levels of net exports in B and I sectors respectively. We assume $X_I^* = 0$ and $X_B^* = 0$ but the results are true for any given X_I^* and X_B^*.

$$p_B^* B^{s*} + I^{s*} = w^* L^* + r^* K^* \text{ (zero profits)}$$

$$K^* = K^s = K_1^* + K_2^* \text{ (capital market clears)}$$

$$L^* = L^s = L^d = L_1^* + L_2^* \text{ ('involuntary' labor market clears)}$$

Notice however that labor market clears in the sense that 'produced' labor supply is equal to labor demand. But since 'produced' labor supply is given by $L^s = L^s(w, p_B)$ and is different from the potential labor supply, N, the labor market is characterized by the existence of involuntary unemployment, since $N > L^*$, as shown in Figure 1. Workers without a job are willing to work at a wage below w^* but will not be hired because below the clearing wage, the L^s produced by these workers is below firm's requirement since workers are not as educated or not as healthy to do the needed job.

We can characterize 'involuntary' labor market equilibrium in a more explicit way. Denoting $l_1 = \frac{L_1}{K_1}$ and $l_2 = \frac{L_2}{K_2}$, we can rewrite marginal conditions (3) as:

$$w = \alpha(l_1)^{\alpha-1} p_B \text{ and } r = (1-\alpha)(l_1)^{\alpha} p_B$$

$$w = \gamma\beta(l_2)^{\beta-1} \text{ and } r = \gamma(1-\beta)(l_2)^{\beta}$$

Indicating logarithms with the symbol "$\sim$" the four equations above can be rewritten as:

$$\widetilde{w} = (\alpha - 1)\widetilde{l_1} + \widetilde{\alpha} + \widetilde{p_B} \text{ and } \widetilde{r} = \alpha\widetilde{l_1} + \widetilde{(1-\alpha)} + \widetilde{p_B} \tag{6}$$

$$\widetilde{w} = (\beta - 1)\widetilde{l_2} + \widetilde{\beta} + \widetilde{\gamma} \text{ and } \widetilde{r} = \beta\widetilde{l_2} + \widetilde{(1-\beta)} + \widetilde{\gamma}$$

so that $(\alpha - 1)\widetilde{l_1} - (\beta - 1)\widetilde{l_2} = \widetilde{\beta} - \widetilde{\alpha} - \widetilde{p_B} + \widetilde{\gamma}$
and $\alpha\widetilde{l_1} - \beta\widetilde{l_2} = \widetilde{(1-\beta)} - \widetilde{(1-\alpha)} - \widetilde{p_B} + \widetilde{\gamma}$.
Solving for $\widetilde{l_1}$ and $\widetilde{l_2}$ we obtain:

$$\widetilde{l_1} = \frac{(\widetilde{\beta} - \widetilde{p_B} - \widetilde{\alpha} + \widetilde{\gamma})(-\beta) - (1-\beta)[\widetilde{(1-\beta)} - \widetilde{p_B} - \widetilde{(1-\alpha)} + \widetilde{\gamma}]}{[\beta - \alpha]}$$

$$\widetilde{l_2} = \frac{(\alpha - 1)[\widetilde{(1-\beta)} - \widetilde{p_B} - \widetilde{(1-\alpha)} + \widetilde{\gamma}] - \alpha(\widetilde{\beta} - \widetilde{p_B} - \widetilde{\alpha} + \widetilde{\gamma})}{[\beta - \alpha]}$$

or

$$\widetilde{l_1} = \frac{\widetilde{p_B}}{(\beta - \alpha)} + A \text{ and } \widetilde{l_2} = \frac{\widetilde{p_B}}{(\beta - \alpha)} + B \tag{7}$$

where A and B such that:

$$A = \frac{(\widetilde{\beta} - \widetilde{\alpha})(-\beta) - (1-\beta)[(\widetilde{1-\beta}) - (\widetilde{1-\alpha})] - \widetilde{\gamma}}{[\beta - \alpha]} \tag{8}$$

$$B = \frac{(\alpha - 1)[(\widetilde{1-\beta}) - (\widetilde{1-\alpha})] - \alpha[\widetilde{\beta} - \widetilde{\alpha}] - \widetilde{\gamma}}{[\beta - \alpha]}$$

A and B are constants since $\widetilde{\gamma}$ is taken as a constant by each individual firm. Also observe that $A > 0$ and $B < 0$ if $\beta < \alpha$. Therefore,

$$l_1 = e^A p_B^{\frac{1}{\beta-\alpha}} \text{ and } l_2 = e^B p_B^{\frac{1}{\beta-\alpha}} \tag{9}$$

Using (9), marginal conditions can be written as:

$$w = \alpha e^{A(\alpha-1)} p_B^{\frac{\beta-1}{\beta-\alpha}} \text{ and } r = (1-\alpha) e^{A\alpha} p_B^{\frac{\beta}{\beta-\alpha}} \tag{10}$$

Substituting (10) into $L^s = L^s(w, p_B)$, we can rewrite L^s as a function of p_B alone, or:

$$L^s = L^s(p_B) \tag{11}$$

'Involuntary' labor market clears, or:

$$L^s(p_B) = L^d(p_B) = L^{*20} \tag{12}$$

Since $l_2 = \frac{L^* - L_1}{K^S - K_1}$ or $L_1 = L^* - l_2(K^S - K_1)$. At same time, $l_1 = \frac{L_1}{K_1}$, so that $L^* - l_2(K^S - K_1) = l_1 K_1$.

The quantity of K and L demanded in the B sector are:

$$K_1 = \frac{L^* - l_2 K^S}{(l_1 - l_2)} \text{ and } L_1 = \frac{l_1}{(l_1 - l_2)}(L^* - l_2 K^S) \tag{13}$$

From (9), (12) and (13) K and L are functions of a single variable p_B:

$$L_1 = \frac{e^A L^*}{(e^A - e^B)} - \frac{e^A e^B K^s p_B^{\frac{1}{\beta-\alpha}}}{(e^A - e^B)} = L_1(p_B) \tag{14}$$

$$K_1 = \frac{L^* p_B^{\frac{1}{\alpha-\beta}}}{(e^A - e^B)} - \frac{e^B K^s}{(e^A - e^B)} = K_1(p_B) \tag{15}$$

Equations (14) and (15) hold for any level of γ. In particular, taking $\gamma = 1$, we denote production of B and I as $\Psi(p_B)$ and $\Phi(p_B)$ respectively. Therefore, from (1), (14) and (15) we obtain the equilibrium level of output as a function of equilibrium price p_B^*:

[20] Notice that this condition is different from the one used in Chichilnisky-Gorbachev paper [13]. Previous paper had full employment of labor and capital and therefore no involuntary unemployment. Here due to the 'produced labor' function (12) there is involuntary unemployment.

$$B^{s*} = [\frac{e^A L^*}{(e^A - e^B)} - \frac{e^A e^B K^s p_B^{\frac{1}{\beta-\alpha}}}{(e^A - e^B)}]^\alpha [\frac{L^* p_B^{\frac{1}{\alpha-\beta}}}{(e^A - e^B)} - \frac{e^B K^s}{(e^A - e^B)}]^{1-\alpha}$$
$$= \Psi(p_B^*)$$

$$I^{s*} = \gamma[L^* - \frac{e^A L^*}{(e^A - e^B)} - \frac{e^A e^B K^s p_B^{\frac{1}{\beta-\alpha}}}{(e^A - e^B)}]^\beta [K^s - \frac{L^* p_B^{\frac{1}{\alpha-\beta}}}{(e^A - e^B)} - \frac{e^B K^s}{(e^A - e^B)}]^{1-\beta}$$
$$= \gamma\Phi(p_B^*) \tag{16}$$

For industry I, (16), does not express output as an explicit function of equilibrium prices alone as we wished, because $\gamma = \gamma(I)$, and $I = I(\gamma, p_B)$. In order to obtain output as explicit functions of equilibrium prices we must therefore find out the equilibrium value of the scale parameter γ^*. This is an additional "fixed point" problem, since γ depends on I, while I depends on γ. We solve this as follows.

The industry I has increasing returns which are external to the firms in this industry, and the parameter γ increases with the level of output of I. We postulated that

$$\gamma = I^\sigma, \text{ where } 0 < \sigma < 1. \tag{17}$$

At an equilibrium, equations (16) and (17) must be simultaneously satisfied, i.e. $\gamma = [\gamma\Phi(p_B)]^\sigma = \gamma^\sigma\Phi(p_B)^\sigma$ or, $\gamma^{1-\sigma} = \Phi(p_B)^\sigma$.

Thus,

$$\gamma = \Phi(p_B)^{\frac{\sigma}{(1-\sigma)}} \tag{18}$$

Therefore at an equilibrium from (16) and (18) we obtain a relation between the outputs of I and p_B:

$$I^{s*}(p_B) = \Phi(p_B^*)^{\frac{1}{(1-\sigma)}} \tag{19}$$

so that

$$I^{s*}(p_B) = \left\{ [L^* - \frac{e^A L^*}{(e^A - e^B)} - \frac{e^A e^B K^s p_B^{\frac{1}{\beta-\alpha}}}{(e^A - e^B)}]^\beta \times [K^s - \frac{L^* p_B^{\frac{1}{\alpha-\beta}}}{(e^A - e^B)} - \frac{e^B K^s}{(e^A - e^B)}]^{1-\beta} \right\}^{\frac{1}{1-\sigma}} \tag{20}$$

The Resolving Equation $F(p_B)$

We can now give explicitly the 'resolving' equation for the model, denoted $F(p_B)$ below:

$$F(p_B) = I^{d*}(p_B) - I^{s*}(p_B) = I^{d*}(p_B) - \Phi(p_B^*)^{\frac{1}{(1-\sigma)}} = 0$$

or,

$$F(p_B) = I^{d*}(p_B) - \left\{ [L^* - \frac{e^A L^*}{(e^A - e^B)} - \frac{e^A e^B K^s p_B^{\frac{1}{\beta-\alpha}}}{(e^A - e^B)}]^{\beta} \times [K^s - \frac{L^* p_B^{\frac{1}{\alpha-\beta}}}{(e^A - e^B)} - \frac{e^B K^s}{(e^A - e^B)}]^{1-\beta} \right\}^{\frac{1}{1-\sigma}} = 0 \quad (21)$$

where from (10) and (4), $I^{d*}(p_B)$ is a function of p_B alone:

$$I^{d*}(p_B) = (\alpha e^{A(\alpha-1)} p_B^{\frac{\beta-1}{\beta-\alpha}}) L^* + ((1-\alpha) e^{A\alpha} p_B^{\frac{\beta}{\beta-\alpha}}) K^s - p_B^* B^{d*}(p_B^*) \quad (22)$$

Solving the equation $F(p_B) = 0$, gives an equilibrium value of p_B^* from which all equilibrium values of other variables (K_1^*, K_2^*, L_1^*, L_2^*, w^*, r^*, B^{s*}, I^{s*}, B^{d*}, I^{d*}, γ^*) can be computed. The model is thus solved.

3 Volatility in the Knowledge Sectors

Our hypothesis is that the level of employment in the increasing returns to scale sectors (IRS) is more volatile than that of other sectors, in the sense that employment increases more in IRS sectors than in the rest of the economy during upturns; during downturns, it contracts more than in the rest of the economy. To test this hypothesis we introduce a measure of volatility of employment, similar to that used in our previous paper, and explore its behavior in a general equilibrium model with involuntary unemployment.

Volatility Index: The 'Labor Beta'

We define an index to measure volatility that is independent of the scale of the variables, denoted 'labor beta'. This is a statistical relative of the financial markets concept of 'beta' that is frequently used to measure volatility of stock prices:

$$\beta = \frac{Cov(X,Y)}{Var(Y)} \quad (23)$$

Here X and Y are employment levels rather than stock prices, X representing employment in a sector and Y, aggregate employment. β_{IRS} denotes the 'labor beta' associated with the increasing returns sector and β_{CRS} that of constant returns to scale sector. Our hypothesis can now be stated as:

Hypothesis: $\beta_{IRS} > \beta_{CRS}$.

Volatility: The Effects of Shocks on Employment Across Sectors

To study employment volatility we assume that there are random shocks to the 'fundamentals' of the model (technologies, preferences, demand, initial endowments of capital and labor). The fundamentals thus vary from period to period, although there are no intertemporal links between one period or another. The equilibrium value of p_B^* varies with the fundamentals producing fluctuations in all equilibrium values of the model, and in particular in equilibrium levels of labor utilized by each sector: L_1^* and L_2^*. Using the general equilibrium model, we study the attendant variations in L_1^* and L_2^*, exploring the extent to which employment in sector I is systematically more volatile than that in sector B.

From the 'resolving' equation, $F(p_B)$, we know how all the variables of the model fluctuate with p_B^*. Expressing labor demand for each sector as a function of output of that sector will allow us to use the main result of the "Volatility in the Knowledge Economy" article[21] to prove that employment volatility of the IRS sector I is larger than that of CRS sector B. Solving the firm's cost minimization problem L_1 and L_2 can be expressed as functions of factor prices and total output of their respective sectors, or:

$$L_1 = B\left(\frac{(1-\alpha)r}{\alpha w}\right)^{\alpha} \text{ and } L_2 = I\left(\frac{(1-\beta)r}{\beta w}\right)^{\beta}$$

From (10), (16), and (19), the above expressions are functions of a single parameter p_B found after solving the 'resolving' equation $F(p_B)$ (21), or

$$L_1(p_B) = \Psi(p_B)\left(\frac{(1-\alpha)r(p_B)}{\alpha w(p_B)}\right)^{\alpha} \text{ and } L_2(p_B) = \Phi(p_B)^{\frac{1}{(1-\sigma)}}\left(\frac{(1-\beta)r(p_B)}{\beta w(p_B)}\right)^{\beta} \tag{24}$$

Proposition 1. *Employment volatility in the increasing returns to scale sector I is larger than that in the constant returns to scale sector B, i.e. the 'labor beta' of external IRS sector, β_{IRS}, is larger than the 'labor beta' of the CRS sector, β_{CRS}, namely $\beta_{IRS} > \beta_{CRS}$, when increasing returns are large enough, i.e.* $0 < \sigma < 1$ *and* $\sigma \sim 1$.

Proof. We aim to prove the following inequality:

$$\beta_{IRS} = \frac{Cov(L_{2t}^*, L_t^*)}{Var(L_t^*)} > \beta_{CRS} = \frac{Cov(L_{1t}^*, L_t^*)}{Var(L_t^*)} \tag{25}$$

where $L_t^* = L_{1t} + L_{2t}$. Because the denominators of (25) are equal and positive, we can restate it as:

[21] Specifically, in that article we showed that as $\sigma \rightarrow 1$, $\beta_{IRS} > \beta_{CRS}$, or that output of the IRS sector is more volatile than that of the CRS sector.

$$Cov(L^*_{2t}, L^*_t) > Cov(L^*_{1t}, L^*_t) \tag{26}$$

Rewriting (26):

$$\sum_{t=1}^{T}(L_{2t} - \overline{L_2})(L^*_t - \overline{L^*}) > \sum_{t=1}^{T}(L_{1t} - \overline{L_1})(L^*_t - \overline{L^*}) \tag{27}$$

where $\overline{L_{1t}}, \overline{L_{2t}}$ and $\overline{L^*_t}$ denote time averages: $\overline{L_1} = \frac{1}{T}\sum_{t=1}^{T}(L_{1t})$.

Rearranging the terms (26) becomes:

$$\sum_{t=1}^{T}\left\{\left[(L_{2t} - \overline{L_2}) - (L_{1t} - \overline{L_1})\right](L^*_t - \overline{L^*})\right\} > 0$$

Substituting for L^*_t and $\overline{L^*}$:

$$\sum_{t=1}^{T}\left\{\left[(L_{2t} - \overline{L_2}) - (L_{1t} - \overline{L_1})\right]\left[(L_{1t} + L_{2t}) - (\overline{L_1 + L_2})\right]\right\} > 0, \text{ or}$$

$$\sum_{t=1}^{T}\{L^2_{2t} - L^2_{1t}\} + T\{(\overline{L_1})^2 - (\overline{L_2})^2\} > 0$$

From (21) and (24) we obtain:

$$\sum_{t=1}^{T}\left\{\Phi(p_B)^{\frac{2}{(1-\sigma)}}\left(\frac{(1-\beta)r(p_B)}{\beta w(p_B)}\right)^{2\beta} - \Psi(p_B)^2\left(\frac{(1-\alpha)r(p_B)}{\alpha w(p_B)}\right)^{2\alpha}\right\}$$
$$+T\{(\overline{L_1})^2 - (\overline{L_2})^2\} > 0 \tag{28}$$

For every t, as $\sigma \to 1$ and $\Phi_t(p_B) > 1$, the inequality (28) is satisfied since the first term $(\Phi_t(p_B)^{\frac{2}{(1-\sigma)}})$ dominates the equation. It follows therefore, that $\beta_{IRS} > \beta_{CRS}$.

4 Empirical Issues

In this section we validate empirically our theoretical result that employment in IRS sectors is more volatile in CRS sectors.

Data Sources and Structure

We use employment data provided by the *Survey of Current Business (SCB)*[22], prepared by the *U.S. Department of Commerce, Bureau of Economic Analysis* (BEA), in Washington, DC, utilizing 'Full Time and Part Time Employees'

[22] Monthly government publications to be found in www.bea.gov.

by industry in thousands for 1948-2001 period (2-3 digit Standard Industry Codes (SIC))[23]. There is a "break" in the time series of this data due to the SIC reclassification. Thus, 1948-1987 data uses 1972 SIC classification, whereas 1987-2001 uses 1987 SIC classification (the estimates of 1977-1987 have not been adjusted to 1987 SIC code due to "lack of adequate data" according to BEA May 2003 publication [34]).

IRS and Traditional Industries

We reviewed the literature and adopted its findings about IRS sectors,[24] as in our 2004 article [13]. We also used a simple correlation between quantities produced and prices charged on the level of industry to identify IRS sectors (the sector is considered to exhibit IRS if correlation is negative)[25]. Figure 3 lists the IRS sectors identified by the empirical literature, showing in each case the respective sources. From the list in Figure 3 we separated out industries that as in the previous paper, we characterize as having internal IRS due to high fixed costs[26]. We were left with 8 industries with *external* economies of scale, which we compared with some of the World Knowledge Competitiveness Index benchmarks (see footnote 2 for details):

1. Credit agencies other than banks (SIC 61)
2. Electronic equipment and instruments (36, 38)
3. Machinery, except electrical (35)

[23] Detailed data series are available at the BEA webpage:
http://www.bea.gov/bea/dn2/gpo.htm

[24] A study by S. Basu and J. Fernald [3] of US private economy (2-3 SIC) find IRS in: 1. Metal Mining, 2. Construction, 3. Furniture, 4. Paper, 5. Primary Metals, 6. Fabricated Metals, 7. Electrical Machinery, 8. Motor Vehicles, 9. Transportation, 10. Communication, 11. Electric Utilities, 12. Wholesale and Retail, 13. Services (various).

Work by W. Antweiler and D. Trefler [1] examine 27 manufacturing and 7 non-manufacturing industries (no services) for 71 countries over 1972-1992 period and identified 11 industries with increasing returns: 1. Petroleum and Coal Products, 2. Pharmaceuticals, 3. Electric and Electronic Machinery, 4. Petroleum Refineries, 5. Iron and Steel Basic Industries, 6. Instruments, 7. Non-Electric Machinery, 8. Forestry, 9. Livestock, 10. Crude Petroleum and Natural Gas, 11. Coal Mining. Their general equilibrium model estimates scale for these industries in the rage of 1.10 to 1.20.

Paul and Siegel [28] find that scale economies are prevalent in US manufacturing. In particular, this study finds evidence of external economies of scale due to supply-side agglomeration.

[25] Specifically, we used chain-type quantity index for GDP by industry and chain-type price index for GDP by industry from BEA for our correlation computations. Detailed data files can be downloaded at: http://www.bea.gov/bea/dn2/gpo.htm.

[26] It is important to carefully study the structure of each industry before characterizing it as having internal or external IRS. This is a subject of our further research in this area.

	Identification
Agriculture, forestry, and fishing	AT
Credit agencies other than banks	corr=<0
Coal mining	AT
Communications	BF
Construction	BF
Electronic equipment and instruments	BF, AT
Fabricated metal products	BF
Furniture and fixtures	BF
Machinery, except electrical	AT
Metal mining	BF
Motor vehicles and equipment	BF
Oil and gas extraction	AT
Paper and allied products	BF
Petroleum and coal products	AT
Primary metal industries	BF
Retail trade	BF
Security and commodity brokers	corr<0
Services	BF
Telephone and telegraph	corr<0
Transportation	BF
Wholesale trade	BF

Source: U.S. Department of Commerce, Bureau of Economic Analysis. Detailed annual series can be found at: http://www.bea.gov/bea/dn2/gpo.htm.

Antweiler and D. Trefler study of 2000. Correlation coefficient is computed as correlation between Total nominal GDP by industry with respect to industry's Price Deflator.

Fig. 3. List of Increasing Returns to Scale Industries Identified by the Literature Review and the Correlation Coefficient Specification

4. Retail Trade (52-59)
5. Security and commodity brokers (62)
6. Services (70, 72-73, 75-76, 78-89)
7. Telephone and telegraph (481, 482, 489)
8. Wholesale Trade (50, 51).

Employment in Increasing Returns Sectors Is More Volatile

Total employment's 'labor beta', by definition, is equal to one. Figure 4 provides 'labor betas' for industries with external economies of scale and for some traditional industries. What is immediately apparent from Figure 4 is that 'labor betas' for IRS industries are on average larger than that of traditional industries, and that the volatility in the service sector drives the results.

In addition, breaking the data into three sub-periods, 1947-1987 and 1988-2001 (chosen arbitrarily at the break of the series and to be consistent with our previous paper) provides another interesting view of the data. The 'labor betas' for IRS on average are largest in the later period, or in 1987-2001 period. This phenomenon could be explained by our model: as $\sigma \rightarrow 1$, 'labor beta' of the IRS sector becomes larger.

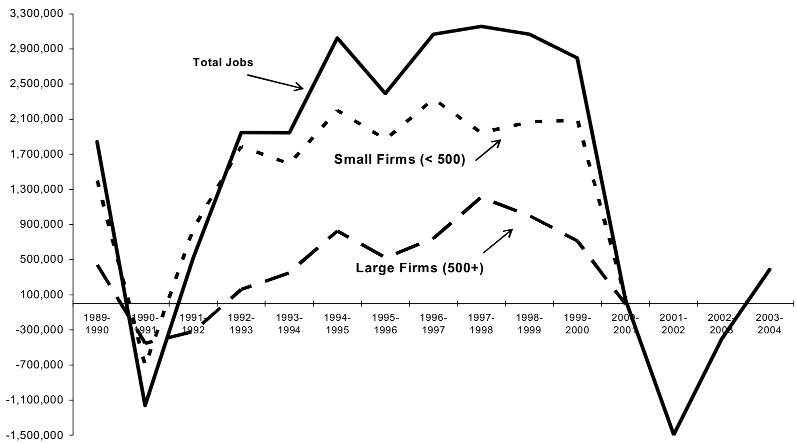
Industry (SIC)	labor beta 1947-2001	labor beta 1947-1987	labor beta 1988-2001	labor shares	% small firms 1997
External IRS					
Wholesale trade (50, 51)	0.06	0.06	0.03	0.05	65
Services (70, 72-73, 75-76, 78-89)	0.44	0.36	0.53	0.22	55
Retail Trade (52-59)	0.18	0.18	0.15	0.15	51
Machinery, except electrical (35)	0.01	0.02	0.00	0.02	47
Electronic equipment and instruments (36, 38)	0.02	0.03	0.00	0.03	43
Credit agencies other than banks (61)	0.01	0.01	0.01	0.00	35
Security and commodity brokers (62)	0.01	0.01	0.01	0.00	28
Telephone and telegraph (481, 482, 489)	0.00	0.01	0.01	0.01	10
Average IRS	*0.26*	*0.22*	*0.29*	*0.48*	*53*
Traditional					
Real estate	0.02	0.02	0.01	0.01	88
Miscellaneous manufacturing industries	0.00	0.00	0.00	0.00	67
Lumber and wood products	0.00	0.00	0.00	0.01	64
Furniture and fixtures	0.00	0.00	0.00	0.01	59
Printing and publishing	0.01	0.01	0.00	0.01	51
Leather and leather products	0.00	0.00	0.00	0.00	42
Textile mill products	-0.01	-0.01	-0.01	0.01	31
Apparel and other textile products	-0.01	0.00	-0.02	0.01	29
Food and kindred products	0.00	0.00	0.00	0.02	28
Paper and allied products	0.00	0.00	0.00	0.01	28
Chemicals and allied products	0.00	0.01	0.00	0.01	21
Average Traditional	*0.00*	*0.00*	*0.00*	*0.11*	*48*

Source: U.S. Department of Commerce, Bureau of Economic Analysis. Detailed annual series on "Full Time and Part Time Employment by Industry" can be found at: www.bea.gov/bea/dn2/gpo.htm; % of small firms within the industry in 1997 is provided by the U.S. Small Business Administration Office of Advocacy, "US Small Business Indicators 2001," 2002.
Note: Shares are averages over 1947-2001 periods and are computed as a number of workers employed in an industry as a share of the total employment. Average IRS and Average Traditional are weighted averages of the series in the respective columns using the provided employment shares.

Fig. 4. Employment Volatility in IRS Sectors is Larger than in Traditional Sectors

5 The Puzzle of the Jobless Recovery

We saw that employment in sectors with increasing returns to scale is on the whole more volatile than in other sectors, in the sense of having higher 'labor betas'. We also saw that within the IRS sectors, service sectors have the highest volatility, or highest 'labor betas'. Service sectors employ 22 percent of the U.S. workers (see Figure 4) and are populated by small firms: 55 percent of the firms in the service industry have less than 500 employees. Small firms generate most new jobs in the US, between 60-80 percent between 1989 and 2001, as illustrated in Figure 5 [40], even though they employ only about 50 percent of the workers. According to the numerous studies by the Small Business Administration, 80 percent of industries that generated most employment (during upturns) and lose most employment (during downturns) over the last 10 years have been in service sector [41]. The service sector is key to our study: on the whole it has increasing returns to scale; it generates most jobs; and it is populated by small firms.



Source: Census Bureau, "County Business Patterns Survey". The total employment data is from January 1989 to March 2004, but employment by size of the firm is from January 1989 until January 2001.

Fig. 5. Net Jobs Generated by Size of Firm

If IRS industries were a large part of the economy, they should, in principle, generate a large increase in employment during a recovery. The US economy grew 4.1 percent in real terms in the fourth quarter of 2003 (surpassing its lowest point of third quarter of 2001) [37]. Yet employment has not yet reached its pre-recession levels. In order for employment to attain its pre-recession levels, 1.5 million jobs lost over 2001-2003 periods[27] must be replaced [39]. Since the IRS sectors are a significant portion of the economy, they employ around 50 percent of the workers, then we should have job creation rather than the observed jobless recovery. How do we resolve this puzzle?

There are at least two possible explanations for the jobless recovery: (i) due to the increased uncertainty in the economy (geopolitical and economic), external IRS sectors that are populated by small firms have been particularly badly hit and thus have been unable to generate new employment; and (ii) there has been a structural change in the economy that shifted the ‘produced

[27] Calculations are done by the authors.

labor' supply curve to the left, making it more difficult to generate new jobs. Both of these situations would lead in our model to a jobless recovery. A combination of these two explanations is at the core of the jobless recovery and is discussed below.

Small Firms and the Service Sector: Heightened Uncertainty and Structural Change

In the current cyclical recovery of 2001-2004, the service sector could have generated most employment, since it has highest 'labor betas' and most new employment comes from this sector [41]. However, the service sector is populated by small firms, which finance their operations through equity and retained earnings rather than debt [42]. Obtaining equity funding or venture capital has been extraordinarily difficult in the environment of high uncertainty[28] both geopolitical and economic since 2001. Equity funding for small firms, as illustrated by Figure 6, fell 86 percent over three year period, 2000-2003[29] [43]. Figure 7 shows that commercial credit, which constitutes 57 percent of small firms' debt also fell by 17 percent. Left without funds to finance their operations, small firms stumbled during this period and therefore generated fewer jobs. Thus, even though the economy was coming out of a recession, the continued uncertainty, especially felt by the service sector small firms, reduced the likelihood of increased job creation generating conditions associated with a jobless recovery (see Figure 8).

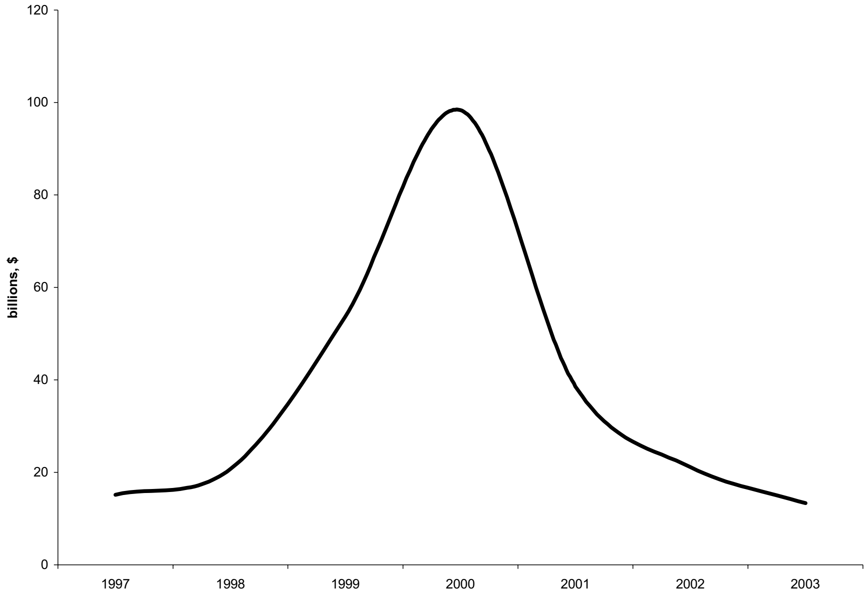
The results of our model support also the Federal Reserve Bank of New York's explanation of the jobless recovery [18]. The FRBNY's explanation for the jobless recovery is that a structural change[30] occurred during the last business cycle. This is supported by their recent study in which the authors show that permanent shifts in the distribution of workers throughout the economy have contributed to the jobless recovery. FRBNY's study proposes three explanations for this change: (i) structural decline might be a reaction to a period of overexpansion, (ii) improved monetary and fiscal policy may have reduced cyclical swings in employment, and (iii) innovation in firm management may be promoting a structural shift towards leaner staffing.

A structural change would require a change in workers' skills. Therefore, even if the recovery created an increased demand for labor as the 'produced' labor function (defined in section 2.03) shifted left, this would have lead to decrease in employment. Figure 9 shows how the new employment level in equilibrium L^{**} could be even less than the previous level L^{*} with such a

[28] Uncertainty is due to (i) economic and political factors, and (ii) to controversy over corporate practices after one of the worse collapses in history of the stock market.

[29] Venture capital fell 61% between 2000 and 2001; 45% between 2001 and 2002; and 37% between 2002 and first half of 2003.

[30] Structural adjustments transform a firm or industry by relocating workers or capital, according to the FRBNY's definition [18].



Source: Small Business Administration Office of Advocacy, "SBIC Program Share of Total Venture Capital Financing to Small Business Reported for Calendar Years 1997-September 2003."

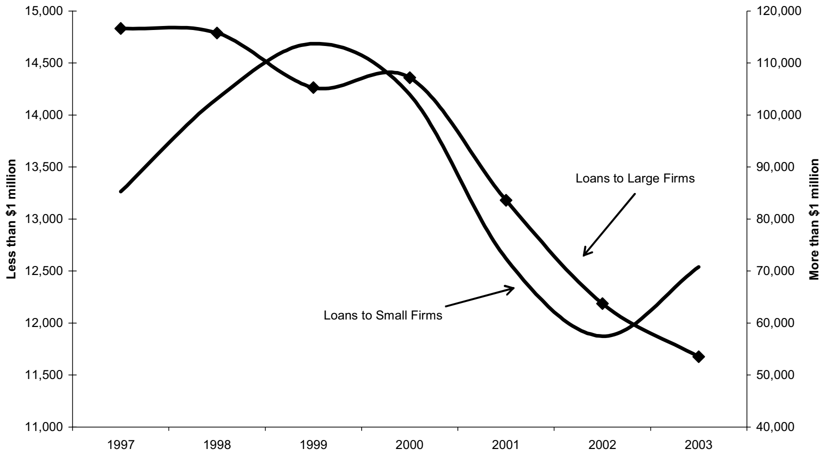
Fig. 6. Venture Capital Financing for Small Firms

shift even with an expansion in labor demand. Our model therefore supports the FRBNY's conclusions.

6 Conclusions and Policies for Job Creation

We showed that employment volatility is higher in the increasing returns to scale sectors than in the rest of the economy. The highest volatility is observed in the service sector, an increasing returns to scale sector mostly populated by small firms with less than 500 employees. It has been shown elsewhere [12] that smaller firms create most new jobs in the economy. Yet small firms have been deprived of funding during the recovery because of heightened uncertainty, geopolitical and economic. This combination of circumstances could be a reason for the jobless recovery.

Indeed, most of small firms' funding comes from equity financing and venture capital that declined 86 percent over the 2001-2003 period. Even though the current US recovery seems to be standard in terms of real output growth, the dramatically reduced funding undermined the ability of small firms to generate employment. Thus we conclude that the first explanation, (i) in section 5.011, for the jobless recovery seems justified.



Source: Federal Reserve Statistical Release, "E.2 Survey of Terms of Business Lending: 1. Commercial and industrial loans made by all commercial banks", 1997-2003. The right hand side axis is for the loans to large firms, i.e. loans that are greater than $1million; and the left hand side axis is for the loans to small firms, i.e. loans that are less than $1million.

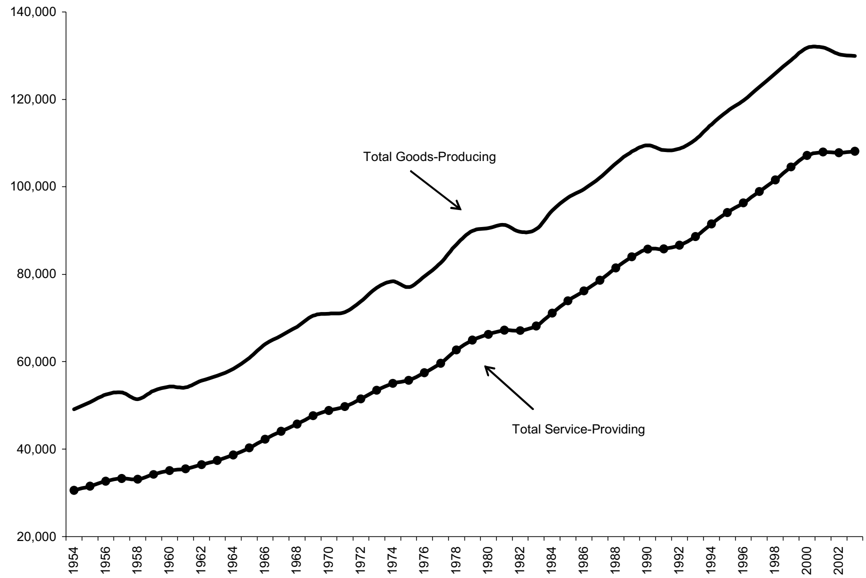
Fig. 7. Lending by All U.S. Banks by Size of Loan, in thousands of dollars

A second feature of our model that helps explain a jobless recovery is the impact of structural change on involuntary unemployment. The Federal Reserve Bank of New York has documented that structural change has occurred in this upswing. This could have shifted the 'produced labor' supply to the left making it more difficult for the economy to create new employment, as in (ii) section 5.011.

The combination of two circumstances (i) the stumbling of small firms, and (ii) structural change, is our preferred explanation for the jobless recovery since 2002.

Several policies can be suggested. Since small firms are viewed as risky, they have limited access to funds, face higher costs of capital, and have less access to top talent. Overcoming these obstacles is what will create most new jobs. Thus, policies should focus on decreasing the risks faced by small firms.

Specific financial policies designed for this purpose were suggested in Chichilnisky [12]. One is to aggregate the equity of smaller firms into 'bundles' that have a lower risk profile due to the 'law of large numbers'. By securitizing these bundles they can be sold in financial institutions – such as stock exchanges – with access to global capital markets. The result is a much larger pool of funds for smaller firms, and lower cost of capital.



Source: U.S. Bureau of Labor Statistics, "Establishment Data Survey: Historical Employment B-1. Employees on Nonfarm Payrolls by Major Industry Sector, 1954 to date."

Fig. 8. Employment Evolution from 1954 to 2003 in Goods-Producing vs. Service-Providing Industries, in thousands

A second proposal is to create credit - enhancement instruments that facilitate the smaller firms' access to capital. A third proposal is to provide access to better borrowing rates for smaller firms by creating large baskets of equity across firms which offer less risk due to diversification – much like Freddie Mae has done successfully in the lower income home mortgage market. Instead of home equity, in this case the assets that back the securities would be the equity in the firms themselves, aggregated throughout many firms in order to reduce risk.

In addition to financial solutions, providing low cost training to employees and benefits to ease sectorial displacements will help resolve problems associated with structural shifts.

Appendix

Workers and the Labor Market (Shapiro and Stiglitz 1984)

Here we provide a specific formulation of the general equilibrium model with involuntary unemployment following analysis of Shapiro and Stiglitz' in "Equilibrium Unemployment as a Worker Discipline Device" [32].

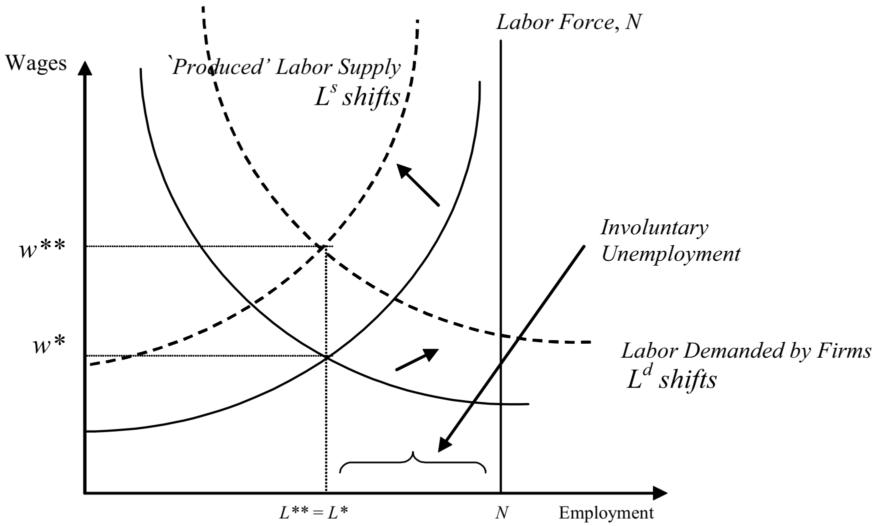


Fig. 9. Structural changes shift the 'Produced Labor' supply curve leading to the Jobless Recovery

There are N identical workers in the economy. All workers dislike putting forth effort, but enjoy consuming goods. Individual's instantaneous utility function can be expressed as $U(c, e)$, assuming for simplicity that the utility function is separable:

$$U(c, e) = H(c) - \Psi(e) \tag{29}$$

where $H(c)$ is a continuous function of workers' consumption bundle, such that $\frac{dH}{dc} > 0$, $\frac{d^2H}{dc^2} < 0$; and $\Psi(e)$ is a continuous function of disutility of effort, e, such that $\frac{d\Psi}{de} > 0$, $\frac{d^2\Psi}{de^2} < 0$.

Each worker maximizes his utility subject to budget constraint:

$$p_c c = wL + rK$$

where p_c is the index of consumer prices and c is the consumption bundle; w stands for wages and r for rental on capital received by each worker.

Assume for simplicity that every employed worker supplies one unit of labor, or $L = 1$ and that utility from consumption has a Cobb-Douglas form. Then, worker's problem can be restated as:

$$\max U(B, I, e) = B^\eta I^{1-\eta} - \Psi(e) \tag{30}$$
$$s.t.\ p_B B + I \leq w + rK$$

where $\eta \in (0, 1)$. By solving (30), demand functions for goods B and I, are given as:

$$B^d = \frac{\eta(wL + rK)}{p_B} = B^d(w, r, p_B) \text{ and } I^d = (1-\eta)(wL + rK) = I^d(w, r, p_B) \tag{31}$$

One can also show that (30) can be rewritten as an indirect utility function

$$V(w, r, e) = \theta(w + rK) - \Psi(e)$$

where $\theta = (\frac{\eta}{p_B})^{1-\eta}(1-\eta)^{\eta}$.

Each worker is in one of two states: employed or unemployed. There is a positive probability b per unit time that a worker will be separated from his job due to unrelated to his productivity causes (i.e. relocation, bankruptcy, etc. of his enterprise), which is exogenously given. Workers maximize the expected present discounted value of utility with a discount rate $\rho > 0$.

$$W = E \int_0^{\infty} V(w_t, r_t, e_t) e^{-\rho t} dt \tag{32}$$

The main choice workers make is the selection of the effort or productivity level. If the worker performs well, or at an expected productivity level, i.e. he doesn't shirk, he receives a wage w with probability $(1-b)$. If the worker shirks, there is a probability $q > 0$ that he will be caught and fired, and therefore will enter the unemployment pool. While unemployed, he receives unemployment compensation $\bar{w}$ set by the government (and assumed to be zero from now on).

The utility from work of a shirker for a short interval $[0, t]$ can be stated as:

$$V_E^S = t\{\theta(w + rK) - \Psi(e)\} + (1 - \rho t)[(bt + qt)V_U + (1 - bt)(1 - qt)V_E^S] \tag{33}$$

where V_U is the expected lifetime utility of an unemployed individual and V_E^S is the expected lifetime utility of an employed shirker. V_E^N is the expected lifetime utility of the employed individual-nonshirker, and can be written as:

$$V_E^N = t\{\theta(w + rK) - \Psi(e)\} + (1 - \rho t)[btV_U + (1 - bt)V_E^N] \tag{34}$$

Solving (34) and (33) by taking a limit of each of the two equations as $t \to 0$, rearranging the terms, taking V_U as given, and assuming for simplicity that $\Psi(e)$ is a discrete function, such that $\Psi(e) = 0$ if the individual shirks, and $\Psi(e) = e > 0$ if he doesn't, we get:

$$\rho V_E^S = \theta(w + rK) + (b + q)[V_U - V_E^S] \tag{35}$$

$$\rho V_E^N = \theta(w + rK) - e + (b)[V_U - V_E^N]$$

or

$$V_E^S = \frac{\theta(w + rK) + (b + q)V_U}{\rho + b + q} \tag{36}$$

$$V_E^N = \frac{\theta(w + rK) - e + bV_U}{\rho + b} \tag{37}$$

The worker will choose to be productive, or to put in $e > 0$ if and only if $V_E^N \geq V_E^S$ or

$$w \geq \frac{\rho V_U}{\theta} - rK + \frac{e}{q\theta}(q + b + \rho) \tag{38}$$

Equation (38) is known as the **No-shirking condition (NSC)**. In order to determine the wage demanded by the workers, we first have to rewrite **NSC** explicitly, taking into account the expected utility of an unemployed worker,Vu. Thus, analogously to equations (35), Vu can be stated as:

$$\rho V_U = \theta(\bar{w} + rK) + a(V_E - V_U) \tag{39}$$

where a is the job acquisition rate and V_E is the expected utility of the employed worker. In equilibrium V_E=V_E^N. Now, plugging (37) into (39) and rearranging the terms, we get:

$$V_U = \frac{\bar{w}\theta(\rho + b) + a\theta w}{\rho(\rho + a + b)} - \frac{ae}{\rho(\rho + a + b)} + \frac{\theta rK}{\rho} \tag{40}$$

Substituting (40) into (38), **NSC** becomes:

$$w \geq \bar{w} + \frac{e}{\theta} + \frac{e}{q\theta}(\rho + b + a)^{31} \tag{41}$$

The implication of **NSC** in this model is that unless there is a penalty from being unemployed (or it takes time before a person can get rehired), there will be no reason for individuals to provide effort[32]. Thus, in order for **NSC** to be satisfied, there will always be a positive number of unemployed.

The wage demanded by the workers positively depends on the job acquisition rate, a. Following the logic of Shapiro-Stiglitz, a can be related to the more fundamental parameters of the model. In steady-state, the flow of workers into the unemployment pool, bL and the flow of workers out of the unemployment pool, $a(N - L)$, (where L is the aggregate employment and N is the total labor force in equilibrium), must be equal:

---

[31] Notice that this **NSC** equation slightly differs from that of Shapiro-Stiglitz. The wage demanded by the workers as in Shapiro-Stiglitz positively depends on the unemployment benefits, effort that the worker puts in, probability of loosing the job b, probability of not getting caught shirking $(1 - q)$, and the job acquisition rate, a. But w depends positively on p_B, since $\frac{\partial\theta}{\partial \bullet_B} = (1-\eta)^{1+\eta}(\frac{\eta}{\bullet_B})^{-\eta}(-\frac{\eta}{\bullet_B^2}) < 0$ and $\frac{\partial w}{\partial\theta} = -\frac{e}{\theta^2}(1 + \frac{1}{q}(\rho + b + a)) < 0$, thus, $\frac{\partial w}{\partial \bullet_B} = \frac{\partial w}{\partial\theta} * \frac{\partial\theta}{\partial \bullet_B} > 0$.

[32] Since NSC can be also stated as $q(V_E^S - V_U) \geq e$, if an individual could immidiately obtain employment after being fired, or $V_E^S = V_U$, or $a = +\infty$, NSC could never be satisfied.

$$a(N-L) = bL \tag{42}$$

Substituting (42) into (41):

$$w \geq \bar{w} + \frac{e}{\theta} + \frac{e}{q\theta}(\frac{bL}{N-L} + \rho + b) \tag{43}$$

Thus, wage demanded is a negative function of the number of unemployed, and as the number of employed (L) increases, wage demanded increases more than proportionately, because it is affected by two simultaneous factors (i) an increase in the number of employed, and (ii) a decrease in the number of unemployed.

Rewriting the above expression we get an explicit labor supply equation:

$$L^s \leq N\frac{eq + e\rho + eb - wq\theta}{eq + e\rho - wq\theta} \tag{44}$$

Or implicitly

$$L^s = L^s(w, p_B, e)$$

such that $\frac{\partial L^s}{\partial w} > 0$[33], $\frac{\partial L^s}{\partial p_B} < 0$[34], and $\frac{\partial L^s}{\partial e} \gtrless 0$ depending on the parameters of the model[35]. Figure 1 illustrates this relationship for a specific effort level e (that could be set to 1 for simplicity).

In equilibrium all markets clear, or:

$$B^{s*} = B^{d*} + X_B^* \text{ (B market clears)} \tag{45}$$

$$I^{s*} = I^{d*} + X_I^* \text{ (I market clears)}$$

where X_B^* and X_I^* are equilibrium levels of net exports in B and I sectors respectively. We assume $X_I^* = 0$ and $X_B^* = 0$ but the results are true for any given X_I^* and X_B^*.

$$p_B^* B^{s*} + I^{s*} = w^* L^* + r^* K^* \text{ (zero profits)}$$

$$K^* = K^s = K_1^* + K_2^* \text{ (capital market clears)}$$

$$L^* = L^s = L^d = L_1^* + L_2^* \text{ (labor market clears)}$$

Notice however that labor market clears in the sense that labor supply is equal to labor demand. But since labor supply is given by the **NSC** (43), the labor market is characterized by the existence of involuntary unemployment, due to

[33] Since $\frac{\partial (L^s)}{\partial w} = q\theta Ne\frac{b}{(eq+e\rho-wq\theta)^2} > 0$.

[34] Since $\frac{\partial L^s}{\partial \theta} = wqNe\frac{b}{(eq+e\rho-wq\theta)^2} > 0$ and $\frac{\partial \theta}{\partial p_B} = (1-a)^{1+a}(\frac{a}{p_B})^{-a}(-\frac{a}{p_B^2}) < 0$, then $\frac{\partial L^s}{\partial p_B} = \frac{\partial L^s}{\partial \theta} * \frac{\partial \theta}{\partial p_B} < 0$.

[35] Since $\frac{\partial (L^s)}{\partial e} = N\frac{eq+e\rho+eb-wq\theta}{(eq+e\rho-wq\theta)e} = \frac{N}{eq+e\rho-wq\theta}q + \frac{N}{eq+e\rho-wq\theta}\rho + \frac{N}{eq+e\rho-wq\theta}b - \frac{N}{(eq+e\rho-wq\theta)e}wq\theta$.

$N > L^*$, as demonstrated by Figure 2. Workers without a job are willing to work at a wage below w^* but cannot make a credible promise not to shirk at such a wage[36], and thus cannot be hired by the firms.

We can characterize labor market equilibrium in a more explicit way. Since the wage paid by each firm must satisfy the **NSC** (43), wages paid by firms given by (10) can be rewritten as:

$$\frac{e}{\theta} + \frac{e}{q\theta}\left(\frac{bL^*}{N - L^*} + \rho + b\right) = \alpha e^{A(\alpha-1)} p_B^{\frac{\beta-1}{\beta-\alpha}}$$

Solving the above expression and taking for simplicity $e = 1$ when workers exert effort, gives us the total labor used in the economy in equilibrium L^* as a function of a single unknown parameter p_B :

$$L^* = N \frac{\left\{\alpha e^{A(\alpha-1)} p_B^{\frac{\beta-1}{\beta-\alpha}} - \frac{1}{\theta}\right\} q\theta - \rho - b}{\left\{\alpha e^{A(\alpha-1)} p_B^{\frac{\beta-1}{\beta-\alpha}} - \frac{1}{\theta}\right\} q\theta - \rho} = L^*(p_B) \tag{46}$$

The rest of the solution of this general equilibrium model follows the main text, writing explicitly the 'resolving' equation for the model:

$$F(p_B) = I^{d*}(p_B) - \left\{ \left[L^* - \frac{e^A L^*}{(e^A - e^B)} - \frac{e^A e^B K^s p_B^{\frac{1}{\beta-\alpha}}}{(e^A - e^B)}\right]^\beta \times \left[K^s - \frac{L^* p_B^{\frac{1}{\alpha-\beta}}}{(e^A - e^B)} - \frac{e^B K^s}{(e^A - e^B)}\right]^{1-\beta} \right\}^{\frac{1}{1-\sigma}} = 0 \tag{47}$$

where from (10) and (12), $I^{d*}(p_B)$ is a function of p_B alone:

$$I^{d*}(p_B) = (1-\eta)\left\{\left(\alpha e^{A(\alpha-1)} p_B^{\frac{\beta-1}{\beta-\alpha}} L^*(p_B) + (1-\alpha) e^{A\alpha} p_B^{\frac{\beta}{\beta-\alpha}} K^*\right)\right\} \tag{48}$$

Solving the equation $F(p_B) = 0$, gives an equilibrium value of p_B^* from which all equilibrium values of other variables (K_1^*, K_2^*, L_1^*, L_2^*, w^*, r^*, B^{s*}, I^{s*}, B^{d*}, I^{d*}, γ^*) can be computed. The model is thus solved.

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Existence of Sunspot Equilibria and Uniqueness of Spot Market Equilibria: The Case of Intrinsically Complete Markets*

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Summary. We consider economies with additively separable utility functions and give conditions for the two-agents case under which the existence of sunspot equilibria is equivalent to the occurrence of the transfer paradox. This equivalence enables us to show that sunspots cannot matter if the initial economy has a unique spot market equilibrium and there are only two commodities or if the economy has a unique equilibrium for all distributions of endowments induced by asset trade. For more than two agents the equivalence breaks and we give an example for sunspot equilibria even though the economy has a unique equilibrium for all distributions of endowments induced by asset trade.

1 Introduction

The purpose of this paper is to clarify the relation between the existence of sunspot equilibria and the uniqueness of spot market equilibria for economies with intrinsically complete asset markets. In these economies Pareto-efficient allocations can be attained as competitive equilibria even without asset trade.

[5] introduce their famous paper "Do sunspots matter?" with the question: "What is the best strategy for playing the stock market? Should one concentrate on *fundamentals* or should one instead focus on the *psychology of the market*". The sunspot literature, originating from [4] contributes to this question by emphasizing that even if all market participants are completely

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rational (in the sense that they are maximizing expected utility functions and have rational expectations), still market outcomes can depend on the psychology of the market. The latter is modelled in the sunspot literature as a publicly observed exogenous random event, nicknamed “sunspot”. Hence, even without referring to any kind of bounded rationality the best way of playing the stock market will not only be based on fundamentals! The simplest case in which this reasoning can be shown to be correct arises when the economic fundamentals allow for multiple equilibria. The sunspot is then used as a coordination device. In this case sunspot equilibria are similar to correlated equilibria studied in the game theory literature. The point of this paper is to analyze whether this is all sunspot equilibria are about, i.e. whether sunspots matter even when spot market equilibria are unique. Throughout the paper we assume that agents are strictly risk averse. Hence if utilities differ across sunspot states then *sunspots matter* because sunspot equilibria are Pareto-inefficient (cf. [5]).

We show that for the case of two agents, with utility functions being concave transformations of additively separable functions, the existence of sunspot equilibria is equivalent to the occurrence of the transfer paradox. As an application of this equivalence we demonstrate that the transfer paradox can occur even if the economic fundamentals can be represented by a *representative consumer*, which in particular implies the uniqueness of equilibria in the initial economy. We also show that the occurrence of sunspot equilibria is indeed subject to the same critique as the occurrence of the transfer paradox. In a model with two commodities, sunspot equilibria can only occur if the initial equilibrium (the equilibrium without asset trade) is not unique. Moreover, if as in the case of Cobb-Douglas economies, uniqueness of equilibria is guaranteed for (almost) all distributions of endowments then sunspot equilibria cannot occur at all. Then we show, again using the equivalence between the two paradoxes, that nevertheless the occurrence of sunspot equilibria does not need to be based on the exogenous selection among multiple equilibria. We construct a simple example in which the equilibrium of any sunspot state is not an equilibrium of any other sunspot state. This example is based on the idea that financial markets may specify incomplete insurance against the uncertainty that they induce. That is to say, in this example asset payoffs are sunspot-dependent and incomplete.

Constructing explicit numerical examples in general equilibrium models is usually done for a class of economies with a simple enough structure so that excess demand functions remain manageable. Computable general equilibrium models (cf. [35]), examples for the occurrence of the transfer paradox ([27, 6, 7, 34, 14], etc), and examples for multiple equilibria ([24, 25, 20, 21]) therefore use the class of CES-utility functions. The class of utility functions assumed in this paper (monotonic transformations of additively separable functions) includes CES-functions. Keeping this assumption we broaden our analysis by looking into the case of more than two agents (countries). We find that the occurrence of the transfer paradox is then no longer sufficient for the existence of sunspot equilibria. Moreover, we give an example with Cobb-Douglas utility functions and sunspot-dependent assets in which sunspots matter even though there are unique equilibria for all distributions of endowments the economy can arrive at using asset trade.

To get the intuition for our results, note that a sunspot equilibrium consists of a set of spot market equilibria (one for each sunspot state) where the endowments of each spot market economy are obtained by redistributions resulting from the equilibrium asset market allocation. Hence an unfavorable redistribution of endowments as it occurs exogenously in the transfer paradox can be derived from asset trade only if the resulting income transfers can be made compatible with the asset allocation the agents choose. In the asset market equilibrium the sum across states (adjusted by the common probability beliefs) of the products of marginal utilities and wealth transfers has to be equal to zero for all agents. Since marginal utilities are non-negative, this requires first of all that all agents find some state with positive income transfer and some with negative income transfer arising from asset trade. Moreover, supposing that marginal utilities are inversely ordered than utility levels, this requires that for some state in which an agent receives a negative income transfer he has a *higher* utility than for some state with a non-negative income transfer. This is achieved by the transfer paradox. In a sense the transfer paradox then "crosses the order" of marginal utility levels across sunspot states. Note that otherwise all states with negative income transfers would be weighted higher than those with positive income transfer and the probability adjusted sum of marginal utilities times income transfers cannot equalize zero. Hence, provided marginal utilities are inversely related to utility levels the transfer paradox is necessary for sunspots to matter. Moreover, by appropriate choice of the agents' risk aversions we can also show the converse, i.e. that a slightly stronger version of the transfer paradox is indeed sufficient for sunspots to matter. The role of assuming concave transformations of additive separable utilities in this reasoning is to guarantee that indeed higher marginal utilities are implied by lower utility levels. This relation is neither true without concavity of the transformation, without additive separability or with more than two agents. In particular, with more than two agents, the existence of sunspot equilibria is no longer tied to the occurrence of the transfer paradox. Of course, a rigorous argument for this intuition will be given once we have made precise the setup of the model considered.

The question concerning the relation of the existence of sunspot equilibria and the uniqueness of spot market equilibria has also been addressed by [29] and [16], for example. One of our results shows that the construction suggested by [29] can indeed be used to show that, using asset trade, an economy can arrive at an endowment distribution for which there are multiple equilibria even though for the initial distribution of endowments equilibria are unique. Note that in [29] it is not shown that "trading towards multiplicity" can be done by starting at an initial distribution of endowments with a unique spot market equilibrium because it is not shown that the equilibrium of the initial economy is unique. Also the Cobb-Douglas example in which sunspot equilibria exist even though spot market equilibria are unique (for all distributions one can arrive at using asset trade) clarifies a confusion that has recently come up in [19] and [3]. [19] has claimed that such an example is possible, but [3] have shown that for the particular parameter values [19] has chosen this is not true! Note that the sunspot equilibrium [3] suggest to consider instead of [19]'s example, sunspots do not matter because agents are not risk averse. Our results in this paper show that the problem in [19] has nothing to do with the parameter values chosen. It arises because [19] considered an economy with only two agents. As we show here, with three agents the construction in [19] is actually possible.

Finally, note that the relation of sunspot equilibria and uniqueness of spot market equilibria has also been studied in economies in which a Pareto-efficient allocation cannot be obtained without asset trade. These types of economies originate in the famous "leading example" of [4]. With first period consumption, asset trade may occur because of intertemporal substitution and as a by-product this may introduce extrinsic uncertainty as [4] has first pointed out.

[16] study an intrinsically incomplete economy a la [4], however with real instead of nominal assets. In our paper assets also have real payoffs but the result of [16], who show the existence of sunspot equilibria for a strong uniqueness assumptions on the underlying economy, does not apply to our setting because it relies on first period consumption. Also for the same reason the *technique* developed by [16] is not applicable here because they control the agents' utility gradients both by perturbing the utility functions and by changing the level of first-period consumption.

In the next section we outline the model. Then we give the definitions of the transfer paradox and of sunspot equilibria. Thereafter we prove our result establishing the equivalence of the transfer paradox and the occurrence of sunspot equilibria. Section 4 applies this result to derive some new insights both for sunspot equilibria and also for the transfer paradox. Also in Section 4 we also show some implications for the existence of sunspot equilibria when spot market equilibria are unique. Section 5 concludes.

2 Model

We first outline the sunspot model. The transfer paradox will then be embedded into this model by a new interpretation of the sunspot states.

There are two periods. In the second period, one of $s = 1, \ldots, S$ states of the world occurs. In the first period assets are traded. Consumption only takes place in the second period. This assumption is important here because otherwise the sunspot model cannot be linked to the atemporal transfer paradox model. There are $i = 1, \ldots, I$ agents and $l = 1, \ldots, L$ commodities in each state. States are called *sunspot states* because the agents' characteristics within the states, i.e. the agents' endowments $\omega^i \in X^i$ and their utility functions $u^i : X^i \to \mathbb{R}$, do not depend on them. X^i is a closed convex subset of $\mathbb{R}^L_+$ which denotes agent i's consumption set. In the sunspot literature the agents' characteristics $[(u^i, \omega^i)_{i=1,\dots,I}]$ are called the *economic fundamentals.* Throughout this paper we make the

Assumption 1 (Additive Separability) *All agents' von Neumann-Morgenstern utility functions u^i are monotonic transformations of additively separable functions, i.e. $u^i(x^i_1, \dots, x^i_L) = f^i(\sum_{l=1}^L g^i_l(x^i_l))$ for all $x^i \in X^i$, where the functions f^i and g^i_l, $l = 1, \dots, L$ are assumed to be twice continuously differentiable, strictly increasing and the g^i_l are concave. Moreover, we assume that for every agent i at least $L-1$ of the functions g^i_l are strictly concave and that for all commodities l there is some i for which g^i_l is strictly concave. Finally, u^i, is assumed to be concave.*

Note that the assumptions on the functions g^i_l guarantee strict quasi-concavity of the function u^i. (Strict-) Concavity of u^i however also depends on the monotonic transformation f^i. The class of utility functions covered by assumption 1 includes all utility functions that are commonly used in applied general equilibrium theory. In particular, the case of CES utilities, $u^i(x^i) = \sum_{l=1}^L \left((\alpha^i_l)^{1-\rho^i}(x^i_l)^{\rho^i}\right)^{1/\rho^i}$, defined for all $i = 1, \dots, I$ on $X^i = \{x \in \mathbb{R}^L_{++} | u^i(x) \geq u^i(\omega^i)\}$, for some $\omega^i \in \mathbb{R}^L_{++}$ and some $0 < \alpha^i_l < 1$, $l = 1, \dots, L$ and $\rho^i < 1$, is covered by this assumption. Also across states agents are assumed to have additive separable utility functions:

Assumption 2 (Expected Utility) *For all agents, $i = 1, \dots, I$, the expected utility functions, defining preferences over state contingent consumption $x^i(s) \in \mathbb{R}^L$, $s = 1, \dots, S$ are given by*

$$\mathbb{E}u^i(x^i(1), \dots, x^i(S)) = \sum_{s=1}^S \pi(s) h^i(u^i(x^i(s))) \quad \forall x^i \in (X^i)^S ,$$

where the h^i are twice continuously differentiable, strictly increasing and concave functions. Moreover, $\mathbb{E}u^i$ is assumed to be strictly concave.

An important subclass of economies arises if the von Neumann-Morgenstern utility functions $h^i(f^i(\sum_l g^i_l(x^i_l)))$ are concave transformations of additively separable functions, i.e. if assumption 1 holds and the composite functions $h^i \circ f^i$ are concave.

Assumption 3 (E.U. with Concave Additive Separability) *Assumption 1 and 2 hold and the composite functions $h^i \circ f^i$ are concave.*

To include CES-functions under assumption 3, the convex transformation $f^i(y) = y^{1/\rho^i}$ has to be transformed by a sufficiently concave function h^i so that $h^i \circ f^i$ is concave. However, to satisfy assumptions 1 and 2 one could choose a strictly concave function that makes the expected utility concave without requiring concavity of $h^i \circ f^i$. Unfortunately, this subtle difference will be important for our paper, as we will give examples with CES-utilities, satisfying assumptions 1 and 2 but not 3, in which sunspot equilibria occur.

In the first period agents can trade $j = 1, \ldots, J$, real assets with payoffs $A^j(s) \in \mathbb{R}^L$ if state s occurs. We denote asset prices by $q \in \mathbb{R}^J$. Agent i's portfolio of assets is denoted by $\theta^i \in \mathbb{R}^J$. Note that all asset payoffs are real, i.e. in terms of commodities. Moreover, we allow for sunspot depended asset payoffs. There is an impressive strand of the sunspot literature originating from [4] in which asset payoffs are nominal. In this literature asset payoffs measured in real terms differ across sunspot states if and only if sunspots matter. The same is effectively also the case in our setting: Supposing spot market equilibria are unique the equilibrium transfers across states measured in real terms depend on sunspots if and only if sunspots matter.

All equilibria we consider in this setting are special cases of competitive equilibria, which are defined in

Definition 1 (Competitive Equilibrium). *A competitive equilibrium is an allocation* $(\overset{\star}{x}{}^i, \overset{\star}{\theta}{}^i)$, $i = 1, .., I$, *and a price system* $(\overset{\star}{p}, \overset{\star}{q})$ *such that*

1. *For all agents* $i = 1, .., I$*:*
   $(\overset{\star}{x}{}^i, \overset{\star}{\theta}{}^i) \in \operatorname{argmax}_{x^i \in X^i, \theta^i \in \mathbb{R}^J} \sum_{s=1}^S \pi(s) h^i(u^i(x^i(s)))$ *s.t.* $\overset{\star}{q} \cdot \theta^i \leq 0$ *and*
   $\overset{\star}{p}(s) \cdot x^i(s) \leq \overset{\star}{p}(s) \cdot \omega^i + \overset{\star}{p}(s) \cdot A(s)\theta^i$ *for all* $s = 1, \ldots, S$.
2. $\sum_{i=1}^I \overset{\star}{x}{}^i(s) = \sum_{i=1}^I \omega^i$ *for all* $s = 1, \ldots, S$.
3. $\sum_{i=1}^I \overset{\star}{\theta}{}^i = 0$.

Remark 1. To simplify the exposition when analyzing competitive equilibrium allocations we restrict attention to *interior* allocations, i.e. to allocations x^i in the interior of X^i, $i = 1, \ldots, I$. A sufficient assumption guaranteeing the interiority of allocations is to impose that the functions f^i and g^i_l satisfy the Inada condition according to which the marginal utility tends to infinity at the boundary of the consumption set $X^i \subset \mathbb{R}^L_+$.

Note that a competitive equilibrium consists of S spot market equilibria (one for each spot market economy with endowments $\hat{\omega}^i(s) = \omega^i + A(s)\theta^i$) together with an asset market equilibrium by which the ex-post endowments of the spot markets are generated. It will be convenient to introduce the spot-market economy of the economic fundamentals as a point of reference. To abbreviate notations we therefore let this economy be the spot market economy in the spot $s = 0$. Finally, note that when showing the existence of sunspot equilibria we allow to choose the characteristics not fixed by the economic fundamentals, the sunspot extension, appropriately. The sunspot extension consist of the probabilities of the sunspot states π, the asset structure A and the risk aversion functions h^i. The sunspot equilibria are robust with respect to perturbations of these characteristics however sunspot equilibria will not exist for all possible choices of the sunspot extension.

This finishes the description of the model.

3 Sunspot Equilibria and the Transfer Paradox

In the sunspot literature agents transfer commodity bundles across sunspot states by trading assets. In the international trade literature one thinks of transfers of commodities arising from donations. Each sunspot state will later on be associated with different such donations. The *transfer paradox* is said to occur if some agent donates some of his resources to some other agent and the recipients utility decreases. In the case of two agents by Pareto-efficiency within spot markets the donor's utility increases. In this statement the utility comparison is done across the competitive equilibria of the economy before and after the donation. In the standard case of the transfer paradox, the transfer was considered to be a transfer of a non-negative amount of commodities ([27]). In order to make the equivalence to the sunspot model more obvious we consider a slightly more general definition of the transfer paradox which only requires that the donated commodities have non-negative *value* in the competitive equilibrium after the transfer. As [14] have already shown this generalization is innocuous.

In the following definition we consider alternative possible transfers $\Delta\omega(z)$ that we index by some scenarios z. When relating the transfer paradox to sunspot equilibria these scenarios will be associated with different states of the world, $s = 1, \dots, S$. Taking care of potentially multiple equilibria the transfer paradox is then defined as in[1]

Definition 2 (Transfer Paradox). *Given an economy with fundamentals* $[(u^i, \omega^i)_{i=1,\dots,I}]$ *the transfer paradox occurs if and only if there exists some transfer of endowments* $\Delta\omega(z) \in \mathbb{R}^{LI}$, *with* $\sum_{i=1}^{I} \Delta\omega^i(z) = 0$ *such that for the economy* $[(u^i, \omega^i + \Delta\omega^i(z))_{i=1,\dots,I}]$ *there exists an equilibrium* $(\overset{\star}{x}(z), \overset{\star}{p}(z))$ *with* $\overset{\star}{p}(z) \cdot \Delta\omega^1(z) \geq 0$ *so that* $u^1(\overset{\star}{x}{}^1(z)) < u^1(\overset{\star}{x}{}^1(0))$ *for some equilibrium* $(\overset{\star}{x}(0), \overset{\star}{p}(0))$ *of the economic fundamentals,* $[(u^i, \omega^i)_{i=1,\dots,I}]$, *in the reference scenario without transfers,* $s = 0$.

Note that under certain conditions and if the economic fundamentals have at least two equilibria then even without any transfers the transfer paradox occurs. Our definition covers this case because then $\Delta\omega = 0$ is already sufficient to obtain $u^1(\overset{\star}{x}{}^1(z)) < u^1(\overset{\star}{x}{}^1(0))$ for the two equilibria $s = 0, z$. Of course if the resulting equilibria are regular then in this case one can also find some

[1] To save on notation we define the transfer paradox with respect to the value of the transfers and changes in utility of agent 1.

transfers of endowments that have positive value and yet the recipients utility decreases. Making the transfer paradox a bit more paradoxical.

We will show that the occurrence of the transfer paradox is a necessary condition for sunspots to matter. To show a converse of this claim we consider the following slightly stronger notion of the transfer paradox.

Definition 3 (Strong Transfer Paradox). *Given an economy with fundamentals* $[(u^i, \omega^i)_{i=1,\ldots,I}]$ *the strong transfer paradox occurs if and only if there exist some transfers of endowments,* $\Delta\omega(z) \in \mathbb{R}^{LI}$*, with* $\sum_{i=1}^{I} \Delta\omega^i(z) = 0$ *and* $\Delta\omega(\tilde{s}) \in \mathbb{R}^{LI}$*, with* $\sum_{i=1}^{I} \Delta\omega^i(\tilde{s}) = 0$ *such that for the economies* $[(u^i, \omega^i + \Delta\omega^i(s))_{i=1,\ldots,I}]$*,* $s = z, \tilde{s}$

1. *there are some equilibria* $(\overset{\star}{x}(z), \overset{\star}{p}(z))$*,* $(\overset{\star}{x}(\tilde{s}), \overset{\star}{p}(\tilde{s}))$ *with* $\overset{\star}{p}(z) \cdot \Delta\omega^1(z) \geq 0$ *and* $\overset{\star}{p}(\tilde{s}) \cdot \Delta\omega^1(\tilde{s}) \leq 0$ *and*
2. *it holds that* $u^1(\overset{\star}{x}{}^1(z)) < u^1(\overset{\star}{x}{}^1(\tilde{s})) < u^1(\overset{\star}{x}{}^1(0))$ *for some equilibrium* $(\overset{\star}{x}(0), \overset{\star}{p}(0))$ *of the economic fundamentals* $[(u^i, \omega^i)_{i=1,\ldots,I}]$*, in the reference scenario without transfers* $s = 0$*.*

Note, that if the economic fundamentals have at least three equilibria then by the same reason as given for the transfer paradox the strong transfer paradox occurs. Hence the existence of at least (three) two equilibria is sufficient for the (strong) transfer paradox. Of course, in regular economies we know that if there are at least two equilibria then there also are at least three equilibria (cf. [9]). This observation indicates that in regular economies the transfer paradox and the strong transfer paradox are actually equivalent. Indeed this it true as the next proposition shows. Recall that in regular economies equilibria are well determined, i.e. in a neighborhood of regular equilibria (being defined by full rank of the Jacobian of market excess demand) there exists a smooth mapping from the exogenous parameters of the economy to the endogenous equilibrium values (cf. [8]). In the following argument regularity needs only be required for the spot market equilibria of the economic fundamentals. This property holds generically in the set of agents' initial endowments $\mathbb{R}^{LI}_{++}$ (cf. [8]).

Proposition 1. *Suppose all spot market equilibria of the economic fundamentals* $[(u^i, \omega^i)_{i=1,\ldots,I}]$ *are regular. Then the transfer paradox and the strong transfer paradox are equivalent.*

Proof

The strong transfer paradox implies the transfer paradox. To establish the converse suppose that the transfer paradox holds. I.e. there exists some transfer of endowments $\Delta\omega(z) \in \mathbb{R}^{LI}$, with $\sum_{i=1}^{I} \Delta\omega^i(z) = 0$, such that for the economy $[(u^i, \omega^i + \Delta\omega^i(z))_{i=1,\ldots,i}]$ there exists an equilibrium $(\overset{\star}{x}(z), \overset{\star}{p}(z))$ with $\overset{\star}{p}(z) \cdot \Delta\omega^1(z) \geq 0$ so that $u^1(\overset{\star}{x}{}^1(z)) < u^1(\overset{\star}{x}{}^1(0))$ for some equilibrium $(\overset{\star}{x}{}^i(0), \overset{\star}{p}(0))$ of the economic fundamentals, $s = 0$.

We need to show that there also exists some $\Delta\omega(\tilde{s}) \in \mathbb{R}^{LI}$, with $\sum_{i=1}^{I} \Delta\omega^i(\tilde{s}) = 0$ such that $\overset{\star}{p}(\tilde{s}) \cdot \Delta\omega^1(\tilde{s}) \leq 0$ and $u^1(\overset{\star}{x}{}^1(z)) < u^1(\overset{\star}{x}{}^1(\tilde{s})) < u^1(\overset{\star}{x}{}^1(0))$.

This is of course the intuitive case in which a negatively valued transfer leads to a loss in utility. However, we need to ensure that this is the outcome in the spot market equilibrium *after* the transfer and that the utility loss is not too severe as compared to the loss in the transfer paradox case. This is ensured by the regularity of the equilibrium of the economic fundamentals from which we construct the transfer appropriately: Consider the utility gradient of agent 1, $\nabla u^1(\overset{\star}{x}{}^1(0))$ at the equilibrium of the economic fundamentals. Choose the transfers $(\Delta\omega^1(\tilde{s}))$, such that $\nabla u^1(\overset{\star}{x}{}^1(0))(\Delta\omega^1(\tilde{s})) < 0$. By the first order condition of utility maximization in the reference situation $s = 0$ we get that this wealth transfer evaluated at the pre-transfer prices is negative, $\overset{\star}{p}(0) \cdot (\Delta\omega(\tilde{s})) < 0$. Since $\nabla u^1(\overset{\star}{x}{}^1(0))(\Delta\omega(\tilde{s})) < 0$, by proposition 31.2 (*ii*) in [28] we can find some $1 \geq \alpha > 0$ such that $u^1(\overset{\star}{x}{}^1(0) + \alpha(\Delta\omega(\tilde{s}))) < u^1(\overset{\star}{x}{}^1(0))$. Moreover, by the regularity of the economy, $\alpha > 0$ can be chosen small enough so that also the utility at the induced equilibrium is smaller than in the reference situation without transfers, $u^1(\overset{\star}{x}{}^1(\tilde{s})) < u^1(\overset{\star}{x}{}^1(0))$. This is because in regular economies the induced change in the equilibrium allocation $\overset{\star}{x}{}^1(\tilde{s})$ can be held small so that $|u^1(\overset{\star}{x}{}^1(\tilde{s})) - u^1(\overset{\star}{x}{}^1(0) + \alpha\Delta\omega(\tilde{s}))|$ is also small. Moreover, by the same continuity argument this can be done such that $\Delta\omega^1(\tilde{s})$ evaluated at prices after the transfer is non-positive, i.e. $\overset{\star}{p}(\tilde{s}) \cdot \Delta\omega^1(\tilde{s}) \leq 0$. Finally, all this can be done without decreasing the utility level too much, so that for agent 1 we get the inequality $u^1(\overset{\star}{x}{}^1(z)) < u^1(\overset{\star}{x}{}^1(\tilde{s})) < u^1(\overset{\star}{x}{}^1(0))$.

□

The strong transfer paradox ensures the order crossing property mentioned in the introduction. To see this note that it is always possible to find transfers of resources, say $\Delta\omega(\hat{s})$, such that the transfer to agent 1 has negative value in the resulting equilibrium, i.e. $\overset{\star}{p}(\hat{s}) \cdot \Delta\omega^1(\hat{s}) \leq 0$, and agent 1 gets a level of utility that is smaller than any of the utility levels considered in the definition of the strong transfer paradox, i.e. $u^1(\overset{\star}{x}{}^1(\hat{s})) < u^1(\overset{\star}{x}{}^1(z)) < u^1(\overset{\star}{x}{}^1(\tilde{s})) < u^1(\overset{\star}{x}{}^1(0))$.[5] By this observation we get three transfers, two with negative value and one with positive value so that the utility decreases for all transfers. As we will see, by assumption 1, in the case of two agents, we then get that the order of the marginal utilities does not coincide with the order or the reverse order of the

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[5]Note that these losses in utility as compared to the equilibrium of the economic fundamentals do not conflict with the fact that trade is voluntary because it may well be that the utility of agent 1 derived from his initial endowments is even smaller than the expected utility obtained in the spot market equilibria. Also the agent is assumed to be a price taker, i.e. he cannot enforce the equilibrium of the economic fundamentals.

transfer values, i.e. the order crossing property is also obtained for marginal utilities.

Note that the transfer paradox concerns the ordering of income transfers and utility levels. In the first order condition for asset demand however *marginal* utilities and not utility levels themselves play a role. Hence we need to know how the levels of marginal utility are related to the utility levels. Keeping prices fixed across different states, by concavity of the utility function, marginal utilities are inversely related to utility levels. This feature occurs for example if the agents have identical and homothetic preferences. In this case however neither sunspot equilibria matter nor the transfer paradox occurs. In general, changes in relative prices induced by redistributions of income are decisive to determine both the level of utility and of marginal utility. It is these changes from which the transfer paradox and also the existence of sunspot equilibria are derived. Nevertheless, with two agents, whose utilities are concave transformations of additively separable functions, we show that marginal utilities are negatively associated to the level of utilities. To make these ideas precise, we first define the agents' indirect utility function and their marginal utility of income within each state without considering the monotonic transformations h^i:
Let

$$v^i(s) = v^i(p(s), b^i(s)) = \max_{x^i \in X^i} f^i\left(\sum_{l=1}^{L} g_l^i(x_l^i(s))\right) \quad s.t. \quad p(s)\cdot x^i(s) \leq b^i(s)$$

be the *indirect utility* of agents i in state s. Since the functions g_l^i, $l = 1, \dots, L$ are concave and since at least $L-1$ of them are strictly concave there is a unique point x^i at which the utility attains its maximum, given that for all commodities the prices $p_l(s)$, $l = 1, \dots, L$ and the income $b^i(s)$ are positive. In our model the income $b^i(s)$ will be given by $p(s)\cdot(\omega^i + A(s)\cdot\theta^i)$. I.e., the values of the transfers are given by $r^i(s) = p(s)\cdot A(s)\cdot\theta^i$. In the analysis of the sunspot model the agents' *marginal utility of income* will be important

$$\lambda^i(s) = \partial_v h^i(v^i(s))\, \partial_b v^i(p(s), b^i(s))\,.$$

Hence the marginal utilities that determine the asset allocation are given by the marginal utilities within each state, $\partial_b v^i(p(s), b^i(s))$, multiplied by the first derivative of the agents' concave transformations h^i determining the agents' risk aversion.

The association between levels of utilities and of marginal utilities is an important link between the transfer paradox and sunspot equilibria which we therefore need to define properly:

Definition 4 (Inverse Association of Utilities and Marginal Utilities). *We say that for some agent i the levels of marginal utility are inversely associated to the levels of utility if*

$$v^i(1) \geq v^i(2) \geq \dots \geq v^i(S)$$

implies

$$\lambda^i(1) \leq \lambda^i(2) \leq \ldots \leq \lambda^i(S) .$$

Moreover, if $v^i(\tilde{s}) < v^i(z)$ *for some* $\tilde{s}, z \in \{1, \ldots, S\}$ *then the corresponding inequality in the marginal utilities of income should also be strict.*

This definition puts us in the position to state the equivalence of the occurrence of the transfer paradox and the existence of sunspot equilibria.

Theorem 1 (Equivalence Sunspot Equilibria and Transfer Paradox). *Suppose assumption 2 holds and all agents' level of marginal utility are inversely associated to their level of utility. Then*

1. *the transfer paradox is a necessary condition for sunspots to matter and*
2. *if there are only two agents then the strong transfer paradox is a sufficient condition for sunspots to matter.*

Proof

1. To link the transfer paradox to the sunspot economy consider

$$r^i(s) := \overset{\star}{p}(s) \cdot A(s)\, \overset{\star}{\theta}{}^i,$$

i.e. the transfer of income to agent i as generated by asset trade in some competitive equilibrium.
A necessary condition for optimal portfolio choice is

$$\sum_{s=1}^{S} \lambda^i(s)\, \pi(s)\, r^i(s) = 0\,, \quad i = 1, \ldots, I\,,$$

which we call the first-order conditions for asset demand.[6]
Now suppose that the transfer paradox does not hold. Then negative transfers $r^i(s) < 0$ are associated with lower utility levels than positive transfers $r^i(s) > 0$ are. If moreover marginal utilities are inversely associated to utilities then negative transfers are associated with higher marginal utility levels than positive transfers are. Hence the first order condition for asset demand requires that the probability weighted sum of the absolute values of negative transfers is smaller than the probability weigthed sum of positive transfers:

$$\sum_{s:r^i(s)>0} \pi(s)\, r^i(s) > - \sum_{s:r^i(s)<0} \pi(s)\, r^i(s)\,, \quad \text{for all } i = 1, .., I\,. \tag{1}$$

This however conflicts with asset market clearing, which implies that income transfers must be balanced:

[6]This condition follows from $\sum_s \lambda^i(s)\, \pi(s)\, p(s)\, A(s) = \gamma^i q$ together with $q \cdot \theta^i = 0$.

$$\sum_{i=1}^{I} r^i(s) = 0\,, \quad \text{for all } s = 1, \ldots, S\,. \tag{2}$$

To see this, multiply equation (2) by $\pi(s)$ and sum those equations over states to obtain:

$$\sum_{s=1}^{S}\sum_{i=1}^{I} \pi(s)\, r^i(s) = 0. \tag{3}$$

Rearranging terms this implies

$$\sum_{i=1}^{I} \left(\sum_{s:r^i(s)>0} \pi(s)\, r^i(s) - \sum_{s:r^i(s)<0} \pi(s)\, r^i(s) \right) = 0\,, \tag{4}$$

which contradicts equation (1).

2. Suppose the strong transfer paradox occurs, then there exist transfers indexed by $\tilde{s}, z$ such that

$$r^1(z) \geq 0\,,\ r^1(\tilde{s}) \leq 0 \quad \text{and for some equilibria} \quad v^1(z) < v^1(\tilde{s}) < v^1(0)$$

where $v^1(0)$ refers to agent 1's utility in an equilibrium of the spot economy $s = 0$. Given the utility functions u^1, u^2 and given the total endowments $\omega^1 + \omega^2$ consider the set of Pareto-efficient allocations as being parameterized by the income transfers r.

Now we have to distinguish three cases:

Case 1: If $r^1(z) > 0$

then we know that $b^1(z) > 0$ and therefore there exists $r^1(\hat{s}) < 0$ sufficiently small such that for the induced $b^1(\hat{s}) = (b^1(z) + r^1(\hat{s})) \geq 0$ we get $v^1(\hat{s}) < v^1(z)$ for some equilibrium in $\hat{s}$. By thisobservation and the strong transfer paradox we have the order crossing property:

$$r^1(\hat{s}) \leq 0\,, \quad r^1(\tilde{s}) \leq 0\,, \quad r^1(z) > 0$$

while

$$v^1(\hat{s}) < v^1(z) < v^1(\tilde{s})$$

so that by the negative association of marginal utilities to the level of utilities

$$\lambda^1(\hat{s}) > \lambda^1(z) > \lambda^1(\tilde{s})\,.$$

To construct the sunspot equilibrium consider an economy with the three states $s = \hat{s}, \tilde{s}, z$. In this case the first-order conditions for asset demand become:

$$\lambda^i(\hat{s})\,\pi(\hat{s})\,|r^i(\hat{s})| + \lambda^i(\tilde{s})\,\pi(\tilde{s})\,|r^i(\tilde{s})| = \lambda^i(z)\,\pi(z)\,|r^i(z)|\,,\ i = 1, 2\,.$$

Now choose $\pi(z) < 1$ sufficiently large (and accordingly $\pi(\hat{s}) > 0$ and $\pi(\tilde{s}) > 0$ sufficiently small) such that

$$\pi(\hat{s})\,|r^i(\hat{s})| + \pi(\tilde{s})\,|r^i(\tilde{s})| < \pi(z)\,|r^i(z)|\,.$$

Note that ∂h^i is any continuous, positive and decreasing function. Recall that, $\lambda^1(\tilde{s}) < \lambda^1(z)$ and that $v^1(\hat{s})$ is the smallest utility level in the three states. Hence we can choose h^1 such that $\lambda^1(\hat{s})$ is sufficiently large to solve the first order condition for $i = 1$. Analogously it follows that $\lambda^2(\hat{s}) < \lambda^2(z)$ and we can choose h^2 such that $\lambda^2(\tilde{s})$ is sufficiently large to solve the first order condition for $i = 2$.
To complete the proof we follow the analogous steps as in [29]. Choose $A \in \mathbb{R}^{3L\times 2}$ such that

$$r^1(s) = p(s)\cdot(A^1(s) - A^2(s)) \quad \text{for } s = \tilde{s}, \hat{s}, z\,. \tag{5}$$

Finally, note that

$$\sum_s \lambda^1(s)\,\pi(s)\,p(s)\cdot A^1(s) = \sum_s \lambda^1(s)\,\pi(s)\,p(s)\cdot A^2(s)$$

so that we can choose $q_1 = q_2$. Accordingly we choose $\theta^1 = (1, -1)$, $\theta^2 = (-1, 1)$ so that $q\cdot\theta^i = 0, i = 1, 2$ and $\theta^1 + \theta^2 = 0$. Since we have chosen an economy with two assets, the first-order conditions for asset trade are equivalent to the conditions $\sum_s \lambda^i(s)\,\pi(s)\,p(s)\,A(s) = \gamma^i q$.

Case 2: If $r^1(z) = 0$ and $r^1(\tilde{s}) = 0$
then by the strong transfer paradox, even without trading any asset, there is a competitive equilibrium in which sunspots matter.

Case 3: Finally, the case $r^1(z) = 0$ and $r^1(\tilde{s}) < 0$
is already covered by the reasoning of the first case if one changes the point of view from agent 1 to agent 2.

□

Remark 2. Note that in the theorem above part 1 has been shown for the most general statement without evoking any particular assumption on the asset structure $A \in \mathbb{R}^{SL\times J}$. part 2 however is a stronger claim the more the set of asset structures can be restricted. The choice of the asset structure matters in equation (5) of the proof. One way of restricting A is to only consider numeraire assets so that all assets pay off in the same commodity. Allowing for sunspot dependent assets this is a possible choice in the solution of equation (5). If assets are not allowed to depend on the sunspot states then one can still find an asset structure solving equation (5), provided the three price vectors $p(s)$, $s = \tilde{s}, \hat{s}, z$ are linearly independent. The latter then requires to have at least 3 commodities.

To complete this section we first show that under assumption 3 in the case of two agents the order of the marginal utilities of income is inverse to the order of their (indirect) utilities. Hence not only in the trivial case of identical and homothetic preferences we get this property but we also get it for all numerical examples with two agents that have so far been considered in the sunspot and in the transfer paradox literature.

Lemma 1. *Suppose there are only two agents. Without loss of generality assume that in a competitive equilibrium*

$$v^1(1) \leq v^1(2) \leq \ldots \leq v^1(S).$$

Then under assumption 3 it follows that

$$\lambda^1(1) \geq \lambda^1(2) \geq \ldots \geq \lambda^1(S)$$

and that

$$\lambda^2(1) \leq \lambda^2(2) \leq \ldots \leq \lambda^2(S)\,.$$

Moreover, if $v^1(\tilde{s}) < v^1(z)$ for some $\tilde{s}, z \in \{1, \ldots, S\}$ then the corresponding inequality in the marginal utilities of income is also strict.

Proof
Assume that

$$v^1(\tilde{s}) \leq v^1(z) \quad (\text{resp. that } v^1(\tilde{s}) < v^1(z)\,) \quad \text{for some } \tilde{s}, z \in \{1, \ldots, S\}\,.$$

Then, by monotonicity of the utility function, for some commodity, say $k \in \{l, \ldots, L\}$ we must have that

$$x^1_k(\tilde{s}) \leq x^1_k(z) \quad (\text{resp. that } x^1_k(\tilde{s}) < x^1_k(z)\,)\,.$$

Moreover, Pareto-efficiency within spot markets implies that for all states $s = 1, \ldots, S$ the marginal rates of substitution are equal across agents, i.e.

$$\frac{\partial g^1_m(x^1_m(s))}{\partial g^1_l(x^1_l(s))} = \frac{\partial g^2_m(x^2_m(s))}{\partial g^2_l(x^2_l(s))}$$

for any pair of commodities (l, m). Note that $x^2_m(s) = \omega^1_m + \omega^2_m - x^1_m(s)$, $s = 1, \ldots, S$. Hence if the functions g^i_l are concave and if for some agent the function g^i_l is strictly concave then it follows that

$$x^1_l(\tilde{s}) \leq x^1_l(z) \quad (\text{resp. that } x^1_l(\tilde{s}) < x^1_l(z)\,) \quad \text{for all} \quad l = 1, \ldots, L\,.$$

Without loss of generality assume that $l = n$ is the numeraire in all states $s = 1, \ldots, S$, where n is chosen such that g^1_n is strictly concave. Hence we have shown that

$$v^1(1) \leq v^1(2) \leq \ldots \leq v^1(S) \quad (\text{with } v^1(\tilde{s}) < v^1(z) \quad \text{for some } \tilde{s}, z\,)$$

implies for the numeraire that

$$x^1_n(1) \leq x^1_n(2) \leq \ldots \leq x^1_n(S) \quad (\text{with } x^1_n(\tilde{s}) < x^1_n(z) \quad \text{for some } \tilde{s}, z\,)\,.$$

From the first order condition to the maximization problem

$$\max_{x^i \in X^i} \sum_l g_l^1(x_l^1(s)) \quad \text{s.t.} \quad p(s)\cdot x^1(s) \leq b^1(s)$$

we get that $\lambda^1(p(s), b^1(s)) = \partial_y(h^1 \circ f^1)(y)\, \partial g_n^1(x_n^1(s))$ for all $s = 1, \ldots, S$. Since $h^1 \circ f^1$, g_n^1 are (strictly) concave and since $x_n^1(s)$ and $y^1(s) = \sum_l g_l^1(x_l^1(s))$ are increasing (resp. strictly increasing) in s we get that

$$\lambda^1(1) \geq \lambda^1(2) \geq \ldots \geq \lambda^1(S) \quad (\text{resp. that } \lambda^1(z) > \lambda^1(\tilde{s})\,).$$

The claim for $i = 2$ follows analogously from the inverse inequalities

$$x_l^2(1) \geq x_l^2(2) \geq \ldots \geq x_l^2(S) \quad \text{for } l = 1, \ldots, L\,,$$

and from

$$v^2(1) \geq v^2(2) \geq \ldots \geq v^2(S)\,,$$

the latter inequalities being implied by Pareto-efficiency within spot markets. □

Before passing to the next section we want to point out that the assumption of additive separability is indeed tight. The inverse association between the levels of marginal utilities and that of utilities, as shown in lemma 1 does not necessarily hold without additive separability. As the following example shows without additive separability one can find that lower utilities are associated with lower marginal utilities. The endowments in this example are supposed to be the ex-post endowments. Hence they are allowed to depend on the sunspot states since the asset payoffs may depend on them.

Remark 3. Consider a two-agent economy with two commodities. The utility functions are[7]:

$$u^1(x^1) = \sqrt{x_1^1 x_2^1} + x_2^1 \quad \text{and} \quad u^2(x^2) = \sqrt{x_1^2 x_2^2} + x_1^2\,.$$

Note that neither of the two utility functions is additively separable but both are strictly monotonically increasing and strictly concave on $\mathbb{R}^2_{++}$ and both satisfy the Inada-conditions. Moreover, note that both utility functions are homogenous of degree one implying that both goods are normal. In situation $s = 1$ the ex-post endowments are

$$\hat{\omega}_1^1(1) = 1\,, \quad \hat{\omega}_2^1(1) = 5 \quad \text{and} \quad \hat{\omega}_1^2(1) = 4\,, \quad \hat{\omega}_2^2(1) = 2\,.$$

There is a unique equilibrium[8] with prices $p(1) = (1,\, 0.7125)$. The equilibrium budgets are:

[7]The transformations $h^i \circ f^i$ are assumed to be the identity, which is a strictly increasing concave function.

[8]All values have been rounded to 4 decimal digits. The exact values can be found at the page `http://www.iew.unizh.ch/home/hens/sunspot`. Uniqueness can be seen from the graph of the excess demand also shown on the webpage.

$$b^1(1) = 4.5623 \quad \text{and} \quad b^2(1) = 5.4249\,.$$

The resulting allocation is:

$$x_1^1(1) = 0.5380\,, \quad x_2^1(1) = 5.6485 \quad \text{and} \quad x_1^2(1) = 4.4620\,, \quad x_2^2(1) = 1.3515\,.$$

The utility levels are:

$$u^1(1) = 7.3917 \quad \text{and} \quad u^2(1) = 6.9177\,.$$

Marginal utilities within state 1, $\left(\partial_b v^i(1) = \frac{v^i(1)}{b^i(1)}\right)_{i=1,2}$, are:

$$\partial_b v^1(1) = 1.6202 \quad \text{and} \quad \partial_b v^2(1) = 1.2752\,.$$

Now consider a second situation $s = 2$ with the same total endowments but with a distribution of ex-post endowments as:

$$\hat{\omega}_1^1(2) = 5\,, \quad \hat{\omega}_2^1(2) = 5 \quad \text{and} \quad \hat{\omega}_1^2(2) = 0\,, \quad \hat{\omega}_2^2(2) = 2\,.$$

Again, there is a unique equilibrium, now with prices $p(2) = (1\,,\ 1.5113)$. The equilibrium budgets are:

$$b^1(2) = 12.5563 \quad \text{and} \quad b^2(2) = 3.0225\,.$$

The resulting allocation is:

$$x_1^1(2) = 2.3164\,, \quad x_2^1(2) = 6.7758 \quad \text{and} \quad x_1^2(2) = 2.6836\,, \quad x_2^2(2) = 0.2242\,.$$

The utility levels are:

$$u^1(2) = 10.7375 \quad \text{and} \quad u^2(2) = 3.4594\,.$$

Marginal utilities within state 2 are:

$$\partial_b v^1(2) = 0.8552 \quad \text{and} \quad \partial_b v^2(2) = 1.1445\,.$$

Note that the second agent's utility **and** his marginal utility has decreased is passing from situation 1 to situation 2. Finally, note that we could also have chosen two strictly concave functions h^i such that the same ordering still holds for the marginal utilities $\lambda^i(s) = \partial_v h^i(v^i)\,\partial_b v^i(s)$.

The next example shows that for more than two agents the strong transfer paradox is no longer sufficient for the existence of sunspot equilibria. The simple reason is that for agent 1 the strong transfer paradox may occur while the two other agents will not find income transfers of opposite sign.

Remark 4. The example to construct the strong transfer paradox is the famous three country example from [6].

There are three agents and two goods. The utility functions are:

$$u^1(x_1^1, x_2^1) = \min(x_1^1, 4.0\, x_2^1)\,,$$
$$u^2(x_1^2, x_2^2) = \min(x_1^2, x_2^2)\,,$$
$$u^3(x_1^3, x_2^3) = \min(2.8\, x_1^3, x_2^3)\,.$$

Note that this example uses Leontief preferences. Hence strictly spoken our assumption 1 is not satisfied. However, these preferences can be attained as a limit case of CES-utility functions. That is to say, perturbing the preferences slightly within the CES-class will establish an example satisfying assumption 1 and if we like to transform it by a sufficiently concave function h^i also assumption 3 can be satisfied, as we mentioned above. Moreover, note that both utility functions, u^i are homogenous of degree one implying that both goods are normal. Consider the situations $s=0, \tilde{s}, z, \hat{s}$ as required by the strong transfer paradox.

Let the matrix of endowments (for both goods per agent and state), with rows corresponding to states $s=0, \hat{s}, z, \tilde{s}$ and with columns corresponding to agents, be:

$$\omega = \begin{pmatrix} (1\,,\ 1.00) & (2\,,\ 1.00) & (1\,,\ 3.00) \\ (1\,,\ 0.10) & (2\,,\ 2.40) & (1\,,\ 2.50) \\ (1\,,\ 1.10) & (2\,,\ 1.00) & (1\,,\ 2.90) \\ (1\,,\ 0.80) & (2\,,\ 1.25) & (1\,,\ 2.95) \end{pmatrix}\,,$$

In all situations there is a unique equilibrium[9]. The equilibrium price vectors are:

$$p(0) = (5.9084\,,\ 1)\,,$$
$$p(\tilde{s}) = (4.4892\,,\ 1)\,,$$
$$p(z) = (9.6382\,,\ 1)\,,$$
$$p(\hat{s}) = (0.9438\,,\ 1)\,.$$

Evaluated at these equilibrium prices the transfers as compared to situation $s = 0$ are:

$$r^1(\tilde{s}) = -0.2\,,\quad r^2(\tilde{s}) = 0.25 \quad \text{and} \quad r^3(\tilde{s}) = -0.5\,,$$
$$r^1(z) = 0.1\,,\quad r^2(z) = 0.00 \quad \text{and} \quad r^3(z) = -0.1\,,$$
$$r^1(\hat{s}) = -0.9\,,\quad r^2(\hat{s}) = 1.40 \quad \text{and} \quad r^3(\hat{s}) = -0.5\,.$$

The equilibrium budgets are:

$$b^1(0) = 6.9084\,,\quad b^2(0) = 12.8169 \quad \text{and} \quad b^3(0) = 8.9084\,,$$
$$b^1(\tilde{s}) = 5.2892\,,\quad b^2(\tilde{s}) = 10.2284 \quad \text{and} \quad b^3(\tilde{s}) = 7.4392\,,$$
$$b^1(z) = 10.7382\,,\quad b^2(z) = 20.2765 \quad \text{and} \quad b^3(z) = 12.5382\,,$$
$$b^1(\hat{s}) = 1.0438\,,\quad b^2(\hat{s}) = 4.2876 \quad \text{and} \quad b^3(\hat{s}) = 3.4438\,.$$

[9] All values have been rounded to 4 decimal digits. The exact values can be found at the page `http://www.iew.unizh.ch/home/hens/sunspot`. Uniqueness can be seen from the graph of the excess demand function which is displayed at the webpage together with the computations.

The resulting allocations are:

$$x = \begin{pmatrix} (1.1218\,, & 0.2804) & (1.8552\,, & 1.8552) & (1.0230\,, & 2.8643) \\ (1.1161\,, & 0.2790) & (1.8634\,, & 1.8634) & (1.0206\,, & 2.8576) \\ (1.0860\,, & 0.2715) & (1.9060\,, & 1.9060) & (1.0080\,, & 2.8225) \\ (0.8744\,, & 0.2186) & (2.2058\,, & 2.2058) & (0.9120\,, & 2.5756) \end{pmatrix},$$

The corresponding utility levels are:

$$\begin{array}{llll} u^1(0) = 1.1218\,, & u^2(0) = 1.8552 & \text{and} & u^3(0) = 2.8643\,, \\ u^1(\tilde{s}) = 1.1161\,, & u^2(\tilde{s}) = 1.8634 & \text{and} & u^3(\tilde{s}) = 2.8576\,, \\ u^1(z) = 1.0860\,, & u^2(z) = 1.9060 & \text{and} & u^3(z) = 2.8225\,, \\ u^1(\hat{s}) = 0.8744\,, & u^2(\hat{s}) = 2.2058 & \text{and} & u^3(\hat{s}) = 2.5756\,. \end{array}$$

Marginal utilities $\left(\partial_b v^i(s) = \frac{v^i(s)}{b^i(s)}\right)_{i=1,2,3.}$ are:

$$\begin{array}{llll} \partial_b v^1(0) = 0.1624\,, & \partial_b v^2(0) = 0.1448 & \text{and} & \partial_b v^3(0) = 0.3215\,, \\ \partial_b v^1(\tilde{s}) = 0.2110\,, & \partial_b v^2(\tilde{s}) = 0.1822 & \text{and} & \partial_b v^3(\tilde{s}) = 0.3841\,, \\ \partial_b v^1(z) = 0.1011\,, & \partial_b v^2(z) = 0.0940 & \text{and} & \partial_b v^3(z) = 0.2251\,, \\ \partial_b v^1(\hat{s}) = 0.8377\,, & \partial_b v^2(\hat{s}) = 0.5145 & \text{and} & \partial_b v^3(\hat{s}) = 0.7479\,. \end{array}$$

Note that the second agent's transfers are never negative while that of the third agent are never positive. Hence these transfers cannot be sustained by asset trade. Finally, note that as compared to situation $s = 0$ the third agent's utility **and** his marginal utility has decreased in passing to situation $s = z$.

4 Sunspot Equilibria and Uniqueness of Spot Market Equilibria

Having established the link between the transfer paradox and the existence of sunspot equilibria we now derive some new results on the existence of sunspot equilibria when spot market equilibria are unique on the one hand and also on the possibility of the transfer paradox on the other hand. Applying part 1 of our theorem and [36] we can rule out sunspot equilibria if marginal utilities are inversely related to utilities and spot market equilibria are unique at all non-negative distributions of endowments. Once again, applying part 1 of our theorem and [2] we can rule out sunspot equilibria in the case of two commodities and two agents if the economic fundamentals do have a unique equilibrium. On the other hand, applying the construction of [29], which is done for the case of two agents, part 2 of our result shows that the transfer paradox can occur even if for the economic fundamentals there exists a representative consumer. We continue, using once more part 2, to show that sunspot equilibria need not be derived from multiple equilibria of the spot market economy that is obtained by asset trade leading to the same endowment distribution in all states. While [29]'s construction uses that there are multiple equilibria for the induced distribution of endowments, this example however exploits the fact that there are multiple equilibria for the initial distribution of endowments. Therefore, we give one more example in which spot market equilibria are unique for all distributions of endowments that can be arrived at by asset trade and still sunspots matter. This final example then settles the question whether sunspot equilibria need to be based on some sort of multiplicity of spot market equilibria.

The following terminology is quite useful. A *randomization equilibrium* is a competitive equilibrium in which for some ex-post endowments the equilibrium allocation in every state s is a spot market allocation for the same economy. If, for example, the economic fundamentals allow for multiple equilibria then there is a randomization equilibrium. [29] has shown that there can also be randomization equilibria if there are multiple equilibria for *some* distribution of endowments that is attainable via asset trade. In both cases the equilibrium allocation of such a competitive equilibrium is a randomization among the set of equilibria of *some* underlying economy. In randomization equilibria sunspots are a device to coordinate agents' expectations. This case of sunspot equilibria has found many applications. In the international trade literature, for example, currency crises are modelled by randomization sunspot equilibria. See, for example, the seminal papers by [31] and [32] and also the interesting empirical papers on this issue by [22] and [23]).

The question that arises from these observations is whether sunspot equilibria could be identified with randomization equilibria. This would then make them very similar to publicly correlated equilibria known in the game theoretic literature ([1])[10]. Hence, the results of this literature would then be applicable to sunspot equilibria.

It is obvious that in our setting with intrinsically complete markets sunspot equilibria necessarily are randomization equilibria if assets are sun-independent assets, if $A(s) = A(1)$, $s = 1, \ldots, S$. It is, however, not obvious at all whether with a general asset structure there can also be sunspot equilibria which are different from randomization equilibria. To clarify this point some more definitions are needed.

Definition 5 (Attainable Endowment Distributions). *Given the economic fundamentals* $[(u^i, \omega^i)_{i=1,\ldots,I}]$ *and given the asset structure* A *the endowment distributions* $\hat{\omega}^i(s)$, $s = 1, \ldots, S$, $i = 1, \ldots, I$ *is attainable if there exists some competitive equilibrium with asset allocation* $(\overset{\star}{\theta}{}^i)$, $i = 1, \ldots, I$, *such that* $\hat{\omega}^i(s) = \omega^i + A(s)\overset{\star}{\theta}{}^i$, $s = 1, \ldots, S$, $i = 1, \ldots, I$.

Based on the attainability concept we now define the uniqueness concept suggested in [29]. This condition has later been called *no potential multiplicity* by [16].

[10] See [12] for relating sunspot equilibria to correlated equilibria in an overlapping generations model.

Definition 6 (Strong Uniqueness). *The economy with the fundamentals* $[(u^i, \omega^i)_{i=1,\ldots,I}]$ *satisfies the strong uniqueness property for some asset structure A, if the spot market equilibria are unique for every attainable endowment distribution.*

Remark 5. With intrinsically incomplete markets sunspots are known to matter even if the economic fundamentals satisfy the strong uniqueness property (cf. [4] and the literature that has emerged from it, [17] and [16]).

As said above, when assets are sun-independent then in our setting with intrinsically complete markets the strong uniqueness property rules out sunspot equilibria. The following corollary to our theorem shows that for general asset structures the strong uniqueness property restricted to non-negative allocations is still sufficient to rule out sunspot equilibria when marginal utilities are inversely related to levels of utility.

Corollary 1. *Under assumptions 1 and 2 and if marginal utilities are inversely related to levels of utility then sunspots do not matter if there are unique spot market equilibria for all non-negative distributions of endowments in the Edgeworth box.*

Proof
Suppose sunspots did matter, then from our main result we know that the transfer paradox needs to occur. However, as [36] has shown, this requires to be able to trade to some non-negative distribution of endowments for which there are multiple equilibria, which is a violation of the strong uniqueness property.

□

Hence applying our lemma 1, in particular in the case of two agents, sunspots do not matter if there are unique spot market equilibria for all distributions of endowments in the Edgeworth box.

However, asking for the strong uniqueness property may be asking too much since not many economic fundamentals will satisfy this property. Hence the question arises whether sunspot equilibria can exist when there are multiple equilibria for *some* distribution of endowments the economy can arrive at using asset trade. As [29] has shown this is definitely the case. However, [29] did not show that this is still the case if for the initial distribution of endowments there is a unique equilibrium. Note if there were also multiple equilibria at that point then sunspots would matter even without evoking asset trade. The question then is whether "trading from uniqueness to multiplicity" is possible. As the following corollary shows, this is not possible if there are two agents and two commodities.

Corollary 2. *Under assumption 3, in the case of two commodities and two consumers sunspots do not matter if the economic fundamentals have a unique equilibrium.*

Proof

Suppose sunspots did matter then from our main result we know that the transfer paradox needs to occur. However, as for example [2][11] has shown, in the case of two commodities this requires to have multiple equilibria for the initial distributions of endowments.

□

From corollary 2 we can see that in the case of two commodities and sun-independent assets it is not possible to "trade from uniqueness to multiplicity". This is because with sun-independent assets sunspots can only matter at distributions of endowments for which there are multiple equilibria.

[19] has claimed that for an economy with two agents and two commodities in which utility functions are concave transformations of Cobb-Douglas utility functions sunspots matter. The corollary 2 shows that this claim is incorrect. Moreover the mistake in [19] cannot be cured by changing the values of the parameters for the same example[12] because that example falls into the broad class of economies which are covered by this note. Indeed for Cobb-Douglas economies the equilibrium at the initial distribution of endowments is unique and the strong uniqueness requirement is satisfied for almost all asset structures A.

Making further assumptions on the agents' risk aversion functions h^i, [33] shows that the construction of [29] does not work if there are only two commodities. Indeed, [33] shows for an economy with an arbitrary number of consumers with additive separable utilities u^i and non-decreasing relative risk aversion, with two commodities and with asset payoffs that are independent of the sunspot states, non-trivial sunspot equilibria do not exist.

The next example shows that with more than two commodities [29]'s construction can be followed to demonstrate that it is well possible to trade from uniqueness to multiplicity. Hence sunspot equilibria do occur even if for the initial distribution of endowments there is a unique equilibrium. By our theorem then also the transfer paradox needs to occur because the economy we construct has two consumers with CES-utility functions. That the transfer paradox arises even if the initial equilibrium is unique should be no surprise since this is well known in that literature. In the example we give we do however have some more structure. For the initial distribution of endowments the excess demand function of the economic fundamentals can be generated by a single representative consumer. Hence our example shows that "the representative consumer is not immune against sunspots" and that even under this strong structural assumption the transfer paradox is possible.

[11] See also the solution to exercise 15.B.10$^{\mathrm{C}}$ from [30] that is given in [18].

[12] This possibility is left open by the observation of [3] who demonstrate that for the specific parameter values chosen in [19] sunspots do not matter!

Corollary 3. *There exist economic fundamentals, satisfying assumptions 1 and 2, with two agents for which there is a representative consumer and yet sunspot equilibria and the transfer paradox occur.*

Proof

The example we give to prove this claim follows the construction of [29] making sure that it can be done even under assumption 1 and 2 and more importantly, that it can be done even for economic fundamentals with a representative consumer:

There are four commodities and two CES-consumers with utility functions:

$$u^i(x^i) = \sum_{l=1}^{4} \left((\alpha_l^i)^{1-\rho^i} (x_l^i)^{\rho^i} \right)^{1/\rho^i}$$

The utility parameters are[13]: $\rho^1 = -250.00000$, $\rho^2 = -250.00000$ and

$$\begin{aligned} \alpha^1 &= (0.24986\,,\ \ 0.24986\,,\ \ 0.25041\,,\ \ 0.24986)\,, \\ \alpha^2 &= (0.25261\,,\ \ 0.25224\,,\ \ 0.24758\,,\ \ 0.24758)\,. \end{aligned}$$

Note that utility functions satisfy assumption 1 and that they are linearly homogenous. Moreover, agents utilities are close to the Leontief case which is convenient because then the Slutzky-subsitution effects contributing to the uniqueness of equilibria are small. Initial endowments are:

$$\begin{aligned} \omega^1 &= (0.01024\,,\ \ 0.01021\,,\ \ 0.01006\,,\ \ 0.0101)\,, \\ \omega^2 &= (0.01011\,,\ \ 0.01009\,,\ \ 0.00994\,,\ \ 0.01)\,. \end{aligned}$$

Note that the second agent's initial endowments multiplied with 1.00971 gives the first agent's endowments, i.e. endowments are collinear and hence, by [11]'s theorem, for this economy there exists a representative consumer.

Suppose there are three sunspot states, $s = 1, 2, 3$. We claim that trade in (sun-independent) assets can be used to arrive at the following distribution of endowments for which there are three spot market equilibria:

$$\begin{aligned} \hat{\omega}^1 &= (0.01414\,,\ -0.01804\,,\ 0.01259\,,\ 0.0197)\,, \\ \hat{\omega}^2 &= (0.00621\,,\ \ \ \ 0.03834\,,\ 0.00741\,,\ 0.00037)\,. \end{aligned}$$

The resulting spot market equilibrium prices are:

$$\begin{aligned} p(1) &= (1\,,\ 1.5801\,,\ \ \ \ 5.05786\,,\ 0.88298)\,, \\ p(2) &= (1\,,\ 1.37985\,,\ \ 0.88617\,,\ 0.35661)\,, \\ p(3) &= (1\,,\ 1.6344\,,\ \ 11.79925\,,\ 3.67601)\,. \end{aligned}$$

Note that the rank of the price matrix is three since the determinant of all prices excluding the numeraire is -11.11. The equilibrium budgets are:

[13] All values have been rounded to 5 decimal digits. The exact values can be found at the page `http://www.iew.unizh.ch/home/hens/sunspot`.

$$
\begin{aligned}
b^1(1) &= 0.06669 \quad \text{and} \quad b^2(1) = 0.10461\,,\\
b^1(2) &= 0.00742 \quad \text{and} \quad b^2(2) = 0.06582\,,\\
b^1(3) &= 0.20558 \quad \text{and} \quad b^2(3) = 0.15770\,.
\end{aligned}
$$

The resulting allocations are:

$$
\begin{aligned}
x^1(1) &= (0.00785\,,\ \ 0.00783\,,\ \ 0.00781\,,\ \ 0.00785)\,,\\
x^2(1) &= (0.01250\,,\ \ 0.01246\,,\ \ 0.01218\,,\ \ 0.01226)\,,
\end{aligned}
$$

$$
\begin{aligned}
x^1(2) &= (0.00205\,,\ \ 0.00204\,,\ \ 0.00205\,,\ \ 0.00206)\,,\\
x^2(2) &= (0.01830\,,\ \ 0.01825\,,\ \ 0.01795\,,\ \ 0.01801)\,,
\end{aligned}
$$

$$
\begin{aligned}
x^1(3) &= (0.01142\,,\ \ 0.01140\,,\ \ 0.01134\,,\ \ 0.01136)\,,\\
x^2(3) &= (0.00893\,,\ \ 0.00890\,,\ \ 0.00866\,,\ \ 0.00871)\,.
\end{aligned}
$$

As was to be expected in a case close to the Leontief-preferences, for both agents and in all states the demand of any good l as a fraction of the sum of all demands stay close to the agents utility parameters α_l^i even though prices vary substantially.

The corresponding utility levels are:

$$
\begin{aligned}
u^1(1) &= 0.03132 \quad \text{and} \quad u^2(1) = 0.04935\,,\\
u^1(2) &= 0.00819 \quad \text{and} \quad u^2(2) = 0.07249\,,\\
u^1(3) &= 0.04544 \quad \text{and} \quad u^2(3) = 0.03513\,.
\end{aligned}
$$

Marginal utilities $\left(\partial_b v^i(s) = \frac{u^i(s)}{b^i(s)}\right)_{i=1,2,3.}$ are:

$$
\begin{aligned}
\partial_b v^1(1) &= 0.46960 \quad \text{and} \quad \partial_b v^2(1) = 0.47178\,,\\
\partial_b v^1(2) &= 1.10458 \quad \text{and} \quad \partial_b v^2(2) = 1.10130\,,\\
\partial_b v^1(3) &= 0.22104 \quad \text{and} \quad \partial_b v^2(3) = 0.22277\,.
\end{aligned}
$$

Note that agent 1's marginal utilities are ordered inversely to his utilities while agent 2's marginal utilities are ordered as his utility levels. This is possible here because, as mentioned above, the assumption 3 is not satisfied for CES-utilities. Evaluating the change in endowments at the new equilibrium prices, the resulting income transfers from agent 1's point of view are:

$$
r^1(1) = 0.00201\,, \quad r^1(2) = 0.02195\,, \quad r^1(3) = -0.05989\,.
$$

Next we argue that for these values we can solve the first order condition for asset demand:

$$
\sum_s h'^i(v^i(s))\,\partial_b v^i(s)\,\pi(s)\,r^i(s) = 0\,,\ \ i = 1, 2\,.
$$

To do so we can still choose the vector of probabilities π and also the degree of the agents' risk aversion given by three values of the function h'^i which are positive and decreasing with v^i. However, the functions h^i are not sufficiently concave to make the composition $h^i \circ f^i$ concave.

The resulting values are $\pi = (0.8\,,\ 0.05\,,\ 0.15)$ for the probabilities and

$$h'^1 = (1.98877\,,\ 2.00032\,,\ 1.97707) \quad \text{and} \quad h'^2 = (1.01074\,,\ 1.00064\,,\ 1.02074)$$

for the risk aversions. Note that these values are smaller the higher the utility levels are.

Since the three spot price vectors are linearly independent, we can now find an asset matrix $A \in \mathrm{IR}^{3\times4\times2}$ such that

$$r(s) = p(s)\cdot(A^1(s) - A^2(s)) \quad \text{for } s = 1,2,3\,. \tag{6}$$

The vector $A^1 - A^2$ we have chosen is:

$$A^1 - A^2 = (-0.00597\,,\ \ 0.02618\,,\ \ -0.00457\,,\ \ -0.01165)\,.$$

Finally, note that

$$\sum_s \lambda^1(s)\,\pi(s)\,p(s)\cdot A^1(s) = \sum_s \lambda^1(s)\,\pi(s)\,p(s)\cdot A^2(s)$$

so that we can choose $q_1 = q_2$. Accordingly we choose $\theta^1 = (1,\,-1)$, $\theta^2 = (-1,\,1)$ so that $q\cdot\theta^i = 0$, $i = 1,2$ and $\theta^1 + \theta^2 = 0$. Since we have chosen an economy with two assets, the first-order conditions for asset trade are equivalent to the conditions $\sum_s \lambda^i(s)\,\pi(s)\,p(s)\cdot A(s) = \gamma^i q$.

Comparing the vector of income transfers and the resulting utility levels we find that agents' utility decrease when they receive positive income transfer, i.e. the transfer paradox occurs even though the order crossing property does not.

$$u^1(\text{no-asset-trade}) = 0.03224 \quad \text{and} \quad u^2(\text{no-asset-trade}) = 0.03242\,.$$

Last but not least, both agents profit from trade, i.e. their reservation utilities are smaller than their utilities when participating in the markets[14]:

$$u^1(\text{no-trade}) = 0.03218 \quad \text{and} \quad u^2(\text{no-trade}) = 0.04831$$

$$\mathbb{E}u^1 = 0.03228 \quad \text{and} \quad \mathbb{E}u^2 = 0.04838\,.$$

□

The next corollary shows that as in the case of intrinsically incomplete markets also with intrinsically complete markets sunspots can still matter even if they do not serve as a coordination device among multiple equilibria.

[14]In the computation of the expected utility we have ignored the functions h^i because they are sufficiently close to the case of risk neutrality. Indeed in this example, the first-order-conditions for asset demand could also be solved without the functions h^i.

Corollary 4. *With sun-dependent assets, under assumption 3, even for the case of two commodities, there are sunspot equilibria which are not randomization equilibria.*

Proof

The example we give to prove this corollary is adapted from the Example 15.B.2 in [30][15]. There are two commodities and two agents with endowments $[(\omega_1^1, \omega_2^1), (\omega_1^2, \omega_2^2)] = [(2, r), (r, 2)]$. Consumption sets are $X^i = \{x \in \mathbb{R}^L_{++} \mid u^i(x) \geq u^i(\omega^i)\}$ and utility functions are given by

$$u^1(x^1) = x_1^1 - \frac{1}{8}(x_2^1)^{-8} \quad \text{and} \quad u^2(x^2) = -\frac{1}{8}(x_1^2)^{-8} + x_2^2 .$$

Aggregate endowments are $\omega = (2 + r,\ 2 + r)$ where $r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}} \approx 0.77$. Figure 1 shows the Edgeworth Box of this economy.[16] The convex curve is

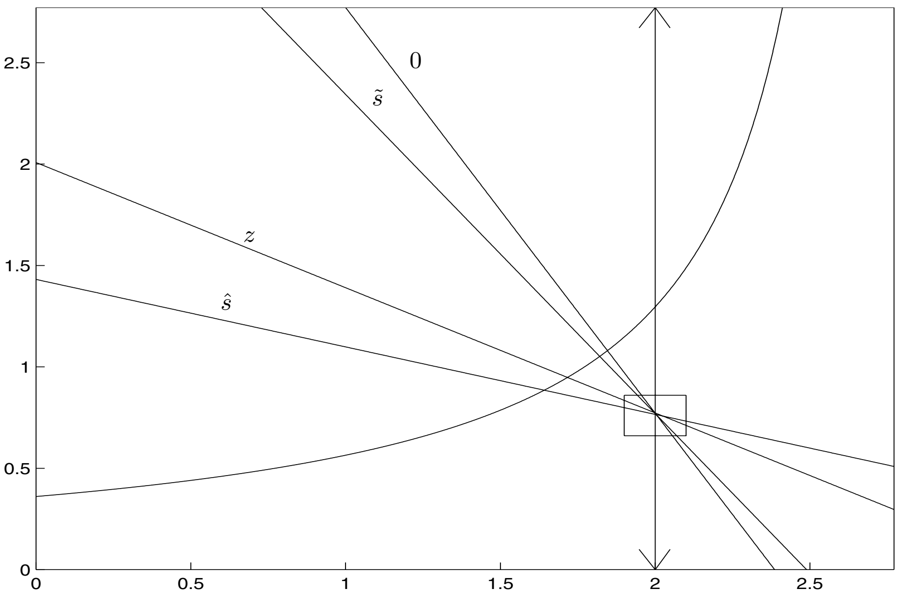

Fig. 1. Edgeworth-Box

the set of Pareto-efficient allocations that lie in the interior of the Edgeworth Box. It is given by the function $x_2^1 = \frac{1}{2+r-x_1^1}$. The competitive equilibrium allocations of our example will be constructed out of these interior allocations. In figure 1 we have also drawn some budget lines indexed by $s = \hat{s}, z, \tilde{s}, 0$,

[15]See [18] for the solution to the original example.

[16]The figures 1 and 2 have been generated with *MATLAB*®. The scripts can be downloaded from the page `http://www.iew.unizh.ch/home/hens`.

supporting four different Pareto-efficient allocations which are equilibrium allocations in the spot markets once appropriate spot market endowments have been chosen. The sunspot equilibrium we construct exploits the fact that in this example there are three equilibria for the distribution of endowments $[(\omega_1^1, \omega_2^1), (\omega_1^2, \omega_2^2)] = [(2, r), (r, 2)]$. Taking these endowments as the reference point for the economy $s = 0$, we consider the transfer of endowments as visualized in figure 2. From the three equilibria at $[(2, r), (r, 2)]$ we have chosen the one with the highest first agent utility to be the equilibrium allocation for the reference situation $s = 0$. The asset structure A consists of numeraire

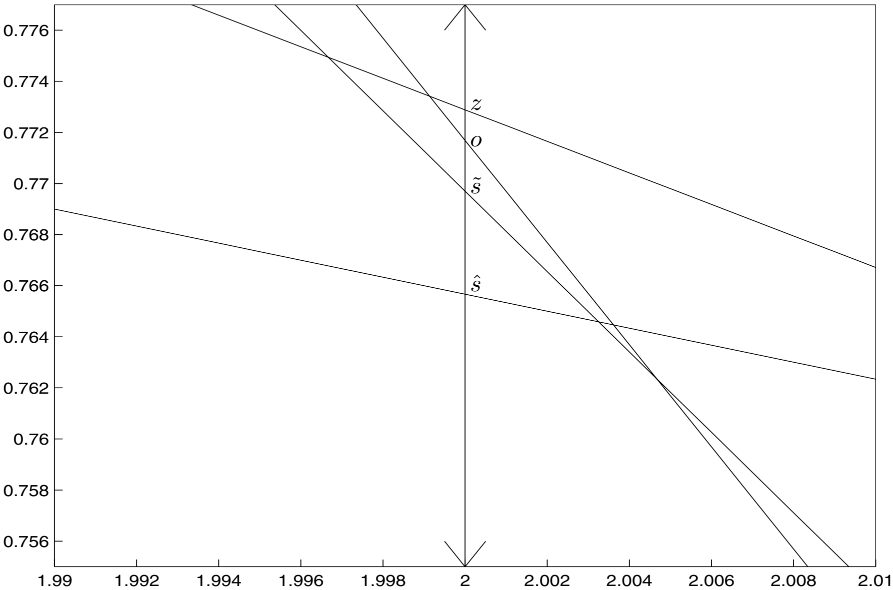

Fig. 2. Zoom of the rectangle in 1

assets denominated in the second commodity. The vertical line in figure 2 indicates the possible direction of endowment redistributions. With reference to $s = 0$, in the situation z the first agent has received a transfer of the second commodity but his utility decreases. In the situations $\hat{s}, \tilde{s}$ the first agent has donated some of the second commodity and with reference to $s = 0$ his utility decreases. While in $\hat{s}$ it falls to the lowest of the four values, in $\tilde{s}$ it obtains a value between the utility in $s = 0$ and $s = z$. Hence for these transfers the strong transfer paradox occurs and by application of our main result there exists a sunspot equilibrium with spot market endowments given by the intersection of the budget lines $\hat{s}, z$, and $\tilde{s}$ with the vertical line through the point $(2, 0)$, while the selected equilibrium in reference economy has the budget line 0.

Although this sunspot equilibrium lies in a neighborhood of a randomization equilibrium it is itself *not* a randomization equilibrium because all spot market endowments differ.

□

This example exploits the multiplicity of equilibria of the economic fundamentals in the sense that in the neighborhood of the endowment distribution leading to multiple equilibria the budget lines have various slopes that are not ordered as the utility levels resulting in the ex post spot market equilibria (see figures 1 and 2). This property could however also occur with a unique equilibrium for the economic fundamentals. Imagine for example that the area in the rectangle of figure 1 would lie outside the Edgeworth-Box. Then keeping the line along which the transfers are defined inside the Edgeworth-Box the same construction could be done. Unfortunately, we could however give no simple utility functions as in Example 15.B.2 in [30] which would lead to this feature.

Our final example shows that sunspot equilibria do exists even if the strong uniqueness property holds. This example follows the construction of [19]. It uses Cobb-Douglas utility functions and sun-dependent assets. As compared to [19] the dimension of the example has been increased as we now use three agents, three commodities, three assets and four states.

Proposition 2. *With sun-dependent assets, and under assumptions 1 and 2, the strong uniqueness assumption does not rule out the existence of sunspot equilibria.*

Proof

There are three commodities and three Cobb-Douglas-consumers with utility functions:

$$u^i(x^i) = \exp \sum_{l=1}^{3} \alpha_l^i \ln(x_l^i)$$

The utility parameters are[17]:

$$\begin{aligned} \alpha^1 &= (0.4072\,,\ \ 0.5112\,,\ \ 0.0816)\,, \\ \alpha^2 &= (0.0269\,,\ \ 0.9301\,,\ \ 0.0430)\,, \\ \alpha^3 &= (0.0088\,,\ \ 0.0019\,,\ \ 0.9892)\,. \end{aligned}$$

Note that utility functions satisfy assumption 1 and that they are linearly homogenous. Initial endowments are:

$$\begin{aligned} \omega^1 &= (0.1093\,,\ \ 0.9045\,,\ \ 0.1008)\,, \\ \omega^2 &= (0.0715\,,\ \ 0.0672\,,\ \ 0.3781)\,, \\ \omega^3 &= (0.8192\,,\ \ 0.0283\,,\ \ 0.5211)\,. \end{aligned}$$

[17] All values have been rounded to 4 decimal digits. The exact values can be found at the page `http://www.iew.unizh.ch/home/hens/sunspot`.

Suppose there are four sunspot states, $s = 1, 2, 3, 4$, which are equally likely to occur. We claim that trade in numeraire but (sun-dependent) assets can be used to arrive at the following distribution of endowments:

Because asset trade only redistributes the first good, to specify the asset structure $A \in \mathbb{R}^{3\times 4\times 3}$, it is sufficient to give the pay offs in the numeraire, good 3:

$$A_3 = \begin{pmatrix} 410.0959 & 726.8543 & 384.2747 \\ -19.8893 & -35.3459 & -18.6341 \\ -386.5250 & -685.5971 & -362.1722 \\ 0.1150 & 0.8172 & 0.0893 \end{pmatrix} .$$

Suppose equilibrium asset prices are $q = (1.0108\,,\ 1.7917\,,\ 0.9472)$ and agent's asset allocation are:

$$\theta = \begin{pmatrix} 0.9911 & -0.1125 & -0.8787 \\ -0.0544 & 0.0777 & -0.0233 \\ -0.9549 & -0.0269 & 0.9818 \end{pmatrix} .$$

In this matrix rows correspond to assets and columns correspond to agents. Then the resulting allocation of good three is:

$$A_3\theta = \begin{pmatrix} 0.1013 & 0.3767 & 0.5220 \\ 0.1032 & 0.3708 & 0.5260 \\ 0.1137 & 0.3385 & 0.5477 \\ 0.0851 & 0.4262 & 0.4887 \end{pmatrix} ,$$

where rows correspond to states and columns correspond to agents.

The induced spot market equilibria have the following price vectors, allocations, values of utilities and of marginal utilities:

$$p = \begin{pmatrix} 0.4358 & 0.9623 & 1 \\ 0.4324 & 0.9518 & 1 \\ 0.4141 & 0.8954 & 1 \\ 0.4638 & 1.0489 & 1 \end{pmatrix} .$$

We give the allocations for each agent in a matrix x^i with rows corresponding to states and columns corresponding to goods:

$$x^1 = \begin{pmatrix} 0.9525 & 0.5415 & 0.0832 \\ 0.9525 & 0.5432 & 0.0825 \\ 0.9527 & 0.5531 & 0.0791 \\ 0.9522 & 0.5285 & 0.0885 \end{pmatrix} ,$$

$$x^2 = \begin{pmatrix} 0.0291 & 0.4567 & 0.0204 \\ 0.0289 & 0.4550 & 0.0200 \\ 0.0278 & 0.4449 & 0.0184 \\ 0.0307 & 0.4698 & 0.0228 \end{pmatrix} ,$$

$$x^3 = \begin{pmatrix} 0.0184 & 0.0018 & 0.8965 \\ 0.0185 & 0.0018 & 0.8974 \\ 0.0194 & 0.0020 & 0.9025 \\ 0.0171 & 0.0017 & 0.8887 \end{pmatrix}.$$

The corresponding levels of indirect utility are:

$$V = \begin{pmatrix} 0.7176 & 0.4248 & 0.8651 \\ 0.7183 & 0.4229 & 0.8661 \\ 0.7225 & 0.4122 & 0.8714 \\ 0.7123 & 0.4388 & 0.8570 \end{pmatrix}.$$

Finally, the levels of marginal utility within states are:

$$\partial_b v = \begin{pmatrix} 0.1121 & 0.1538 & 0.2478 \\ 0.1729 & 0.1537 & 0.2478 \\ 0.1075 & 0.1527 & 0.2476 \\ 0.1177 & 0.1531 & 0.2480 \end{pmatrix}.$$

The resulting income transfers are:

$$A_3\theta = \begin{pmatrix} 0.0004 & -0.0013 & 0.0009 \\ 0.0024 & -0.0073 & 0.0049 \\ 0.0129 & -0.0396 & 0.0266 \\ -0.0157 & 0.0481 & -0.0324 \end{pmatrix}.$$

Note that the transfer paradox does not occur because higher transfers are always associated with higher levels of utility. However, this raises no problem here to the existence of sunspot equilibria because higher utility levels are not always associated with lower marginal utilities!

To complete the argument we need to solve the first order conditions for asset demand

$$\sum_s h'^i(v^i(s))\, \partial_b v^i(s)\, r^i(s) = 0\,, \;\; i = 1, 2\,,$$

by appropriate choice of the degree of the agents' risk aversion:

$$h' = \begin{pmatrix} 0.6414 & 0.6519 & 0.9985 \\ 0.9980 & 0.6583 & 0.9985 \\ 0.6513 & 0.6930 & 0.9983 \\ 0.6282 & 0.5979 & 0.9986 \end{pmatrix}.$$

Note that agents are risk averse since smaller values of $h'^i(s)$ are associated with higher values of utility. However, the degree of risk aversion changes in a non-trivial way with the level of utility. Those changes in risk aversion are know to be very important for asset demand. See the related papers by [20, 21, 15] and also the monograph by [10] for an extensive study.

□

5 Conclusion

Throughout the paper we restricted attention to economies with additively separable utility functions since those functions are commonly used when applying general equilibrium models. We showed that for the case of two agents (countries) the existence of sunspot equilibria is equivalent to the occurrence of the transfer paradox. This equivalence enabled us to show that sunspots cannot matter if the economy has a unique equilibrium for all distributions of endowments induced by asset trade or if the initial economy has a unique spot market equilibrium and there are only two commodities. Then we gave an explicit example to show that with more than two commodities it is actually possible to trade from an endowment distribution leading to a unique equilibrium towards one for which there are multiple equilibria. Moreover, we gave two examples showing that sunspot equilibria need not result from multiplicity of spot market equilibria. In one of those examples, however, there are multiple equilibria at the initial distribution of endowments. In the other example there are sunspot equilibria even though the economy has a unique equilibrium for all distributions of endowments induced by asset trade. This latter example is possible because with more than two agents the existence of sunspot equilibria is no longer tied to the occurrence of the transfer paradox.

We hope that this paper has clarified the relation between the existence of sunspot equilibria and the uniqueness of spot market equilibria for economies with intrinsically complete asset markets. Moreover, further exploring the connection between sunspot equilibria and the transfer paradox, future research might also show interesting new results in related settings. For example one could try to get new insights for the sunspot literature by exploring the results on the transfer paradox in the overlapping generations model ([13]). And similarly there might be new results in storage analyzing economies with transaction costs respectively tariffs ([26]).

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Survival, Uncertainty, and Equilibrium Theory: An Exposition

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Summary. The last twenty years have witnessed a significant growth of the literature on the "survival problem" , primarily in the context of the causes and remedies of famines. Once a subject essentially of empirical development economics, economic survival became an issue of analytical economics and, most recently, of general equilibrium theory. We review several issues of the survival problem in the general equilibrium framework. First, we consider a Walrasian economy with intrinsic uncertainty affecting the endowments and compute the asymptotic probability of survival under different assumptions of dependence among the agents. Next, we show how the presence of extrinsic uncertainty in a dynamic economy may lead to a ruin in a sunspot equilibrium. Finally, we analyze the link between survival and specialization in production.

Key words: Survival, General Equilibrium, Sunspot Equilibrium.

1 Introduction: A Historical Note

In his assessment of the important developments of general equilibrium theory in early fifties, T.C. Koopmans ([20], p.59) observed that "most authors have ignored the analytical difficulty of formulating a model that ensures the possibility of survival, blithely admitting any nonnegative rates of consumption as sustainable. Arrow and Debreu face this issue and find it to be a complicated one..." Koopmans gave a verbal description/interpretation of the Arrow-Debreu assumptions (see Section 4 of [2]) that guaranteed the existence of equilibrium, noting that "they assume that each consumer can, if necessary, survive on the basis of the resources he holds and the direct use of his own labor, without engaging in exchange, and still have something to spare of some type of labor which is sure to meet with a positive price in any equilibrium. If, contrary to the authors' indications, their model were given a stationary state interpretation, it would be found best suited for describing a society of self-sufficient farmers who do a little trading on the side. In modern society few of us can indeed survive without engaging in exchange..." Koopmans felt that "there is considerable challenge to further research on the survival problem" and did touch upon some directions and interpretations somewhat informally, and did also recognize the "inadequacies of any model unable to recognize the element of uncertainty in individual survival."

We note that in his celebrated article Nikaido ([26], p.136) assumed that in his exchange economy the initial endowment vector of each agent was strictly positive. However, in his subsequent note [27], he recognized that this was a "rather strict assumption" in his method of proof (relying upon what came to be known as the Debreu–Gale–Nikaido lemma and generated a substantial literature) of the existence of Walrasian equilibrium. Nikaido proceeded to present a crisp and beautiful extension of the existence theorem to the case where the initial endowment vector was positive ("non-zero, non-negative bundle") and the sum of the initial endowment vectors (over all agents) was strictly positive. Concerned with the application of the model to international trade theory , he felt (quite justifiably) that this weaker assumption was "reasonable enough."

For brevity, we touch upon just one more landmark in equilibrium theory. In Debreu's *Theory of Value* [9], the word "survival" did not figure at all, but some remarks relevant for the interpretations and applications of the Walrasian model are worth recalling. First, he provided a treatment of "uncertainty" in which "the contract for the transfer of a commodity ... specifies,in addition to its physical properties, its location and its date, an event on the occurrence of which the transfer is conditional. This .. definition of a commodity allows one to obtain a theory of uncertainty free from any probability concept and formally identical with the theory of certainty...". This treatment of "uncertainty" postulating the existence of a complete set of markets for all "commodities" originated in Arrow's remarkable paper [1]. With such a definition of "commodities" it is clear that the technology or production set Y_j of producer j and the consumption set X_i of consumer i, are "in general, contained in a coordinate subspace of R^l with a relatively small number of dimensions" ([9] p.38 and p.51). Secondly, Debreu (p.50) asserted that the consumption set X_i of all the possible consumption plans reflected "a priori constraints (for example, of a physiological nature)" (a "concrete" example appears on p.51 with diagrams illustrating consumption sets):

> "... the decision for an individual to have during the next year as sole input one pound of rice and as output one thousand hours of some type of labor could not be carried out".

Another example (p.52) concerns consumption at two dates, with a minimal level of consumption at date 1 which allows the consumer to survive until date 2. In this example, by choosing x_i the consumer "implicitly" chooses "his *life span*". Finally, in his existence theorem on pp.83–84 Debreu assumed (assumption (c) on p.84) that "there is some $x_i^0 \in X_i$ such that $x_i^0 \ll \omega_i$". In his Notes following the text of Chapter 5 (on pp.88–89), he indicated (referring to the contributions of Mckenzie and David Gale) how the assumption (c) could be replaced with other weaker(but, in our view, much less transparent) assumptions involving the interior of the asymptotic cone of the aggregate production possibility set. These assumptions play an essential role in establishing the continuity of the budget set correspondence which is a step towards getting the upper semicontinuity of the excess demand correspondence to which the Debreu–Gale–Nikaido lemma is used, and also in ensuring that the wealth of each agent is positive in equilibrium. Newman [25] has commented on the proper interpretation of the consumption possibility set and the difficulties in handling the survival problem.

It is fair to say that survival was not a serious theme in the subsequent progress of Walrasian equilibrium analysis. However, the importance of this problem was stressed in several prominent publications by Amartya Sen, primarily in the context of his study of famines. Sen [32] quoted Koopmans' remark on the Arrow–Debreu model (that we noted above) and felt that "the problem that is thus eliminated by assumption in these general equilibrium models is precisely the one central to a theory of survival and famines." He commented further that " 'the survival problem' for general equilibrium models calls for a solution not in terms of a clever assumption that eliminates it irrespective of realism, but for a reflection of the real guarantees that actually prevent starvation deaths in advanced capitalist economies." The entitlement approach developed by Sen provided the foundation for modern theoretical analysis of famines. However, it is difficult to capture the sweep of this approach by using formal models, particularly when uncertainty has to be explicitly modeled (see [8] for a deterministic analysis). Sen emphasized that for a better understanding of the problems of survival, one must recognize: (i) for a consumer to survive, his wealth at the equilibrium price system must allow him to obtain the basic necessities, and (ii) "starvation can develop for a group of people as its endowment vector collapses, and there are indeed many accounts of such endowment declines on the part of sections of poor rural population in developing countries... but starvation can also develop with *unchanged* asset ownership through movement of exchange entitlement mapping" ([33], pp.47–48).

Among the themes in the subsequent literature on famines, we note the following:

(1) famines often occur without substantial decline in aggregate food availability:

> "$\cdots$ starvation is a matter of some people not *having* enough food to eat, and not a matter of there *being* not enough food to eat. While the latter can be a cause of the former, it is clearly one of many possible influences." [32]

An example is the Bangladesh famine of 1974, where the availability of food per head, including food production and net imports, in 1974 was higher than in any other year during 1971–76 (see [12]). Thus, a partial equilibrium analysis focusing on the food market may be unable to capture the complexity of events leading to an entitlement failure, and may render misleading policy prescriptions.

(2) the impact of famines may differ for distinct groups of population, in particular, different occupational groups. For example, during the Ethiopian famine of 1972–74 agricultural community suffered the most, and within this group nomadic herdsmen were hit the hardest. The famine itself was initiated by droughts, which resulted in reduced food supply. However,the herdsmen "were affected not merely by the drought but also by the growth of commercial agriculture, displacing some of these communities from their traditional dry-weather grazing land, thereby vastly heightening the impact of the drought"[33]. Other occupational categories that were in the destitution groups included farm servants, rural laborers, craftsmen, women in service occupations. One can see that the most suffering groups were the ones *who did not have command over food production, and whose own production or labor, being less essential for survival, was no longer in demand when food supply dropped.*

> "The characteristics of exchange relations between the pastoral and agricultural economies contributed to the starvation of the herdsmen by making price movements reinforce – rather than counteract – the decline in the livestock quantities. The pastoralist, hit by the drought, was decimated by the market mechanism." ([33], p.112)

Another example is the Bangladesh famine of 1974, during which the families of rural laborers suffered the most, even without decline in the aggregate food availability. The reason in this case was the loss of employment as a result of floods: floods prevented planting of the rice, which "would reduce the food output later, but its impact on employment was immediate and vicious" [12]. To understand this aspect of famines one has to either study the economy at a *disaggregated* level with specialized occupations/endowments or allow for some commodities to play essential role in survival;

(3) expectations about future food prices can play a significant role in market behavior and result in food deprivation of certain groups of population. According to the study by Ravallion [28], a sudden increase in rice prices in Bangladesh during the 1974 famine could be explained by "excessive hoarding" by stockholders, who overestimated the damage to the future crop from floods, and, subsequently, have over-optimistic price expectations. To study the role of expectations one needs to turn to dynamic general equilibrium framework.

A formal approach to the analysis of these themes seems to call for general equilibrium models with specific structures (or, in Lindbeck's words, "concrete" Walrasian systems) and with an explicit recognition of uncertainty. In this paper we review some attempts in this direction.

First, using the earlier works of Hildenbrand ([18], [19]) and Bhattacharya and Majumdar [3] an attempt was made in [4] to define the probability of survival in the presence of *intrinsic uncertainty* affecting the endowments. It is reviewed in Section 2 with an improvement of one of the principal results on the asymptotic behavior of the equilibrium price as the size of the economy increases. It is shown (see (16) that "ruin" may occur either as a result of a collapse of endowments or as a result of an unfavorable movement in the price system. For the case of a weak correlation in endowments and the case of dependence in the form of dependency neighborhoods, the asymptotic results are similar for the ones for the case of independent agents. The issue of modeling a group of agents exposed to a common shock leads to the study of exchangeable random variables, and the problem of characterizing the probability of ruin in a large economy becomes more subtle.

Nest, we turn to the possible role of *extrinsic uncertainty* (a topic to which David Cass has made notable contributions) and give an example of ruin in a sunspot equilibrium. Using a model with overlapping generations and constant endowments we show that there can exist multiple equilibria with inter-generational trade, in some of which all agents survive, and in others old agents are ruined, with all fundamentals being the same. These multiple equilibria are supported by self-fulfilling beliefs, and the agents co-ordinate their beliefs using "sunspots" as a co-ordinating device. Next we turn to a model which links the survival problem to specialization. The main result, in line with [10], is that in an economy with specialization in production a group of agents, whose produce is less essential for survival, is more vulnerable to starvation.

2 The Probability of Survival: Intrinsic Uncertainty in the Cobb–Douglas–Sen Economy

In this and the next section we introduce a survival problem in the Walrasian framework. For simplicity of exposition, we assume that the preferences of the agents can be represented by a Cobb–Douglas utility function. This assumption enables us to compute the Walrasian equilibrium explicitly (see (17): such a computation can be extended to more than two goods). A more general treatment, using Hildenbrand's path-breaking work, of *some* of the issues can be found in [4], although the "language" becomes unavoidably technical. Our exposition focuses on the central economic issues with a rather minimal set of techniques from probability theory, and, hopefully, has pedagogical value.

In what follows, R_{++} is the set of positive real numbers, $x = (x_k) \in R^l$ is non-negative (written $x \geq 0$) if $x_k \geq 0$ for all k, and x is strictly positive (written $x \gg 0$) if $x \in R{+}{+}$. Define $x > 0$ as positive (non-negative and non-zero).

Consider, first, a deterministic Walrasian exchange economy with two goods. Assume that an agent i has an initial endowment $e_i = (e_{i1}, e_{i2}) \gg 0$, and a Cobb–Douglas utility function

$$u(x_{i1}, x_{i2}) = x_{i1}^{\gamma} x_{i2}^{1-\gamma} \tag{1}$$

where $0 < \gamma < 1$ and the pair (x_{i1}, x_{i2}) denotes the quantities of goods 1 and 2 consumed by agent i. Thus an agent i is described by a pair $\alpha_i = (\gamma, e_i)$.

Let $p > 0$ be the price of the first good. In a Walrasian model with two goods, we can normalize prices so that $(p, 1-p)$ is the vector of prices *accepted by all the agents*. The typical agent solves the following maximization problem (P):

$$\text{maximize } u(x_{i1}, x_{i2}) \tag{2}$$

subject to the "budget constraint" defined as

$$px_{i1} + (1-p)x_{i2} = w_i(p)$$

where the income or wealth w_i of the i-th agent is defined as the value of its endowment computed at $(p, 1-p)$:

$$w_i(p) = pe_{i1} + (1-p)e_{i2}. \tag{3}$$

Solving the problem (P) one obtains the *excess* demand for the first good as:

$$\zeta_{i1}(p, 1-p) = [(1-p)/p]\gamma e_{i2} - (1-\gamma)e_{i1} \tag{4}$$

One can verify that

$$p\zeta_{i1}(p, 1-p) + (1-p)\zeta_{i2}(p, 1-p) = 0 \tag{5}$$

The *total excess demand* for the first good at the prices $(p, 1-p)$ in a Walrasian exchange economy with n agents is given by:

$$\zeta_1(p, 1-p) = \sum_{i=1}^{n} \zeta_{i1}(p, 1-p) \tag{6}$$

In view of (5) it also follows that

$$p\zeta_1(p, 1-p) + (1-p)\zeta_2(p, 1-p) = 0 \tag{7}$$

The "market clearing" Walrasian equilibrium price system is defined by

$$\zeta_1(p^*, 1-p^*) = \zeta_2(p^*, 1-p^*) = 0 \tag{8}$$

and direct computation gives us the equilibrium price p_n^* (we emphasize the dependence of equilibrium price on the number of agents by writing p_n^*) as:

$$p_n^* = \left[\sum_{i=1}^{n} X_i\right] / \left[\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i\right] \tag{9}$$

where

$$X_i = \gamma e_{i2},\ Y_i = (1-\gamma)e_{i1}. \tag{10}$$

To be sure, one can verify directly that demand equals supply in the market for the second good when the excess demand for the first good is zero.

Finally, let us stress that a Walrasian economy is "informationally decentralized" in the sense that agent i has no information about (e_j) for $i \neq j$. Thus it is *not* possible for agent i to compute the equilibrium price p_n^*.

One of the suggestions made by Sen ([33], Appendix A) to deal with survival explicitly is now recalled using our notation. Let F_i be a (nonempty) closed subset of R_{++}^2. We interpret F_i as the set of all combinations of the two goods that enable the i-th agent to survive. Now, given a price system $(p, 1-p)$, one can define a function $m_i(p)$ as

$$m_i(p) = \min_{(x_{i1}, x_{i2}) \in F_i} \{p x_{i1} + (1-p) x_{i2}\} \tag{11}$$

Thus, $m_i(p)$ is readily interpreted as the minimum expenditure needed for survival at prices $(p, 1-p)$.

Example: Let $(a_{i1}, a_{i2}) \gg 0$ be a fixed element of R_{++}^2. Let

$$F_i = \{(x_{i1}, x_{i2}) \in R_{++}^2 : x_{i1} \geq a_{i1}, x_{i2} \geq a_{i2}\} \tag{12}$$

Here $m_i(p) = p a_{i1} + (1-p) a_{i2}$.

In our approach we do not deal with the set F_i explicitly. Instead, let us suppose that, in addition to its utility function and endowment vector, each agent i is characterized by a continuous function $m_i(p) : [0,1] \to R_{++}$, and say that for an agent to *survive* at prices $(p, 1-p)$, its wealth $w_i(p)$ (see (3) must exceed $m_i(p)$. Hence, the i-th agent *fails to survive* (or, *is ruined*) at the Walrasian equilibrium $(p_n^*, 1-p_n^*)$ if

$$[p_n^* e_{i1} + (1-p_n^*) e_{i2}] \leq m_i(p_n^*) \tag{13}$$

or, using the definition (3)

$$w_i(p_n^*) \leq m_i(p_n^*) \tag{14}$$

From (13) and (14) one can see that an agent may face ruin due to (a) a possible endowment failure or (b) the equilibrium price system adversely affecting its wealth relative to the minimum expenditure. This issue is linked to the literature on the "price" and "welfare" effects of a change in the endowment on a deterministic Walrasian equilibrium (see the review of the transfer problem by Majumdar and Mitra [22]).

Consider now a case of *intrinsic uncertainty*: suppose that the endowments $\mathbf{e}_i$ of the agents $(i = 1, 2, \cdots n)$ are random variables. In other words, each $\mathbf{e}_i$ is a (measurable) mapping from a probability space$(\Omega, \mathcal{F}, P)$ into the nonnegative orthant of R^2. One interprets Ω as the set of all possible states of environment, and $\mathbf{e}_i(\omega)$ is the endowment of agent i in the particular state ω. The distribution of $\mathbf{e}_i(\cdot)$ is denoted by μ_i [formally each μ_i is a probability measure on the Borel σ field of R^2, its support being a nonempty subset of the strictly positive orthant of R^2]. From the expression (9), the "market clearing" equilibrium price $p_n^*(\omega)$ is random, i.e., depends on ω:

$$p_n^*(\omega) = \left[\sum_{i=1}^{n} \gamma e_{i2}(\omega)\right] / \left[\sum_{i=1}^{n} \gamma e_{i2}(\omega) + \sum_{i=1}^{n} (1-\gamma)\, e_{i1}(\omega)\right] \tag{15}$$

The wealth $w_i(p_n^*(\omega))$ of agent i at $p_n^*(\omega)$ is simply $p_n^*(\omega)e_{i1}(\omega) + [1 - p_n^*(\omega)]e_{i2}(\omega)$. The event

$$\mathcal{R}_n^i = \{\omega \in \Omega : w_i(p_n^*(\omega)) \leq m_i(p_n^*(\omega))\} \tag{16}$$

is the set of all states of the environment in which agent i does not survive. Again, from the definition of the event $\mathcal{R}_n^i$ it is clear that an agent may be ruined due to a meager endowment vector in a particular state of environment. In what follows, we shall refer to this situation as a "direct" effect of endowment uncertainty or as an "individual" risk of ruin. But it is also possible for ruin to occur through an unfavorable movement of the equilibrium prices (terms of trade) even when there is no change (or perhaps an increase!) in the endowment vector. A Walrasian equilibrium price system reflects the entire pattern of endowments that emerges in a particular state of the environment. Given the role of the price system in determining the wealth of an agent and the minimum expenditure needed for survival, this possibility of ruin through adverse terms of trade can be viewed as an "indirect" ("terms of trade") effect of endowment uncertainty.

Our first task is to characterize $P(\mathcal{R}_n^i)$ when n is large (so that the assumption that an individual agent accepts market prices as given is realistic). While this task is certainly made easier by the structure of the model that allows us to compute $p_n^*(\omega)$ explicitly (15), the convergence arguments are still somewhat technical, in particular when we attempt to dispense with the assumption of stochastic independence.

To begin with let us make the following assumptions:

A1. $\{X_i\}$, $\{Y_i\}$ are uniformly bounded[3]: there exists $M > 0$ such that $0 \leq X_i < M$, $0 \leq Y_i < M$ for $i = 1, 2, \cdots$.
A2. $\{X_i\}$ are uncorrelated, $\{Y_i\}$ are uncorrelated.
A3. $\left[(1/n)\sum_i EX_i\right]$ converges to some $\pi_1 > 0$, $\left[(1/n)\sum_i EY_i\right]$ converges to some $\pi_2 > 0$ as n tends to infinity.

In the special case when the distributions of $\mathbf{e}_i$ are the same for all i (so that $1/n\sum_i EX_i = \pi_1$, where π_1 is the common expectation of all X_i; similarly for π_2), A3 is satisfied.

[3] Recall that X_i and Y_i are non-negative by non-negativity assumption on endowments.

Under A1–A3, if the number n of agents increases to infinity, as a consequence of the strong law of large numbers we have the following limiting property of equilibrium prices p_n^*:

Proposition 1. *Under A1–A3, as n tends to infinity, $p_n^*(\omega)$ converges with probability* 1 *(almost surely) to the constant*

$$p_0 = \pi_1 / [\pi_1 + \pi_2] \tag{17}$$

For the proof, we apply Corollary 6.2 in Bhattacharya and Waymire [5] (p.649) to the sequences $\{\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i))\}$ and $\{\frac{1}{n}\sum_{i=1}^{n}(Y_i - E(Y_i))\}$ and conclude that each of these sequences converges to zero with probability 1. Therefore, $\{\frac{1}{n}\sum_{i=1}^{n}X_i\}$ converges with probability 1 to π_1, and $\{\frac{1}{n}\sum_{i=1}^{n}Y_i\}$ converges with probability 1 to π_2 (see, for example, Rohatgi [31], p. 252, Theorem 13). Next, we consider $g(x,y) \equiv x/(x+y)$, a continuous function of a vector (x,y) on R_{++}. By definition of almost sure convergence, $(X_n, Y_n) \overset{a.s.}{\rightarrow} (X, Y)$ implies $g(X_n, Y_n) \overset{a.s.}{\rightarrow} g(X, Y)$ (see, for example, Ferguson [14], p.9). Therefore, $p_n^* \overset{a.s.}{\rightarrow} p_0$. Q.E.D.

Roughly, one interprets (17) as follows: for large values of n, the equilibrium price will not vary much from one state of the environment to another, and will be insensitive to the exact value of n, the number of agents.

In [17] $p_n(\omega)$ was shown to converge *only in probability* to p_0 under weaker assumptions. In our context, the boundedness assumption A1 seems quite innocuous. Since we do not assume stochastic independence, the proof relies on a relatively recent version of the Strong Law of Large Numbers due to Etemadi [13] (see [5] for further discussion).

For the constant p_0 defined by (17), we have the following characterization of the probability of ruin in a large Walrasian economy:

Proposition 2. *If $p_0 e_{i1}(\omega) + (1 - p_0)e_{i2}(\omega)$ has a continuous distribution function,*

$$\lim_{n\to\infty} \left[P(\mathcal{R}_n^i)\right] = P\{\omega : p_0 e_{i1}(\omega) + (1 - p_0)e_{i2}(\omega) \le m_i(p_0)\} \tag{18}$$

Remark: The probability on the right side of (18) does not depend on n, and is determined by μ_i, a characteristic of agent i, and p_0.

2.1 Dependence

Case studies of famines often indicate that a famine is typically confined to a particular geographic region or affects people belonging to the same occupation group (see, for example, [11]). To account for this property one needs to introduce stochastic dependence among the agents in the model. Of course, the most difficult question is how to model the dependence among agents. Also, dependence among random variables complicates asymptotic theory, and, to obtain analytic results, one has to assume a particular structure of the form of dependence. In this section we consider few examples of stochastic dependence among agents in which the limiting results can be derived explicitly. Results parallel to Proposition 1 were obtained in [4] and [16]. Stated informally, three interesting models were tractable: a model involving "weak" correlation among agents ([4], Proposition 1.3); a model with appropriate restrictions on the size of the dependency neighborhood (a concept introduced by Stein [35] and studied by Hashimzade [16] in the present context), and a model with exchangeable (conditionally independent) agents. The last model is particularly important in recognizing that the terms of trade effect may remain significant even in a large economy. We stress the importance of this point with a precise statement of the basic result proved in [4].

2.2 Exchangeability (Exposure to a Common Shock)

To capture the probability of exposure to a common shock to endowments in a simple manner, let us say that μ and ν are two possible probability laws of $\{\mathbf{e}_i(\cdot)\}_{i\geq 1}$. Think of Nature conducting an experiment with two outcomes "H" and "T" with probabilities $(\theta, 1-\theta), 0 < \theta < 1$. Conditionally, given that "H" shows up, the sequence $\{\mathbf{e}_i(\cdot)\}_{i\geq 1}$ is independent and identically distributed with common distribution μ. On the other hand, conditionally given that "T" shows up, the sequence $\{\mathbf{e}_i(\cdot)\}_{i\geq 1}$ is independent and identically distributed with common distribution ν. Let $\pi_{1\mu}$ and $\pi_{1\nu}$ be the expected values of X_1 under μ and ν respectively. Similarly, let $\pi_{2\mu}$ and $\pi_{2\nu}$ be the expected values of Y_1 under μ and ν. It follows that $p_n(\cdot)$ converges to $p_0(\cdot)$ almost surely, where $p_0(\cdot) = \pi_{1\mu}/[\pi_{1\mu}+\pi_{2\mu}] = p_{0\mu}$ with probability θ and $p_0(\cdot) = \pi_{1\nu}/[\pi_{1\nu}+\pi_{2\nu}] = p_{0\nu}$ with probability $1-\theta$. We now have a precise characterization of the probabilities of ruin as n tends to infinity. To state it, write

$$J = \{(u_1, u_2) \in R_+^2 : p_{0\mu}u_1 + (1-p_{0\mu})u_2 \leq m_i(p_{0\mu})\};$$
$$r_i(\mu) = \int_J \mu(du_1, du_2). \tag{19}$$

Similarly, define $r_i(\nu)$ obtained on replacing μ by ν in (19).

Proposition 4. *Assume that $p_0 e_{i1}(\omega) + (1-p_0)e_{i2}(\omega)$ had a continuous distribution function under each distribution μ and ν of $\mathbf{e}_i = (e_{i1}, e_{i2})$.*

(a) *Then, as the number of agents n goes to infinity, the probability of ruin of the i-th agent converges to $r_i(\mu)$, with probability θ, when "H" occurs and to $r_i(\nu)$, with probability $1-\theta$ when "T" occurs.*
(b) *The overall, or unconditional, probability of ruin converges to*

$$\theta r_i(\mu) + (1-\theta) r_i(\nu).$$

Here, the precise limit distribution is slightly more complicated, but the important distinction from the case of independence (or, "near independence") is that the limit depends not just on the individual uncertainties captured by the distributions μ and ν of an agent's endowments, but also on θ that retains an influence on the distribution of prices even with large n.

3 Survival and Extrinsic Uncertainty: An Example With Overlapping Generations

We now turn to *extrinsic* uncertainty: when the uncertainty affects the *beliefs* of the agents (for example, the agents believe that market prices depend on some "sunspots") but the fundamentals are the same in all states. Clearly, with respect to the probability of survival, the extrinsic uncertainty has no direct effect, because it does not affect the endowments. However, it may have an indirect effect: self-fulfilling beliefs of the agents regarding market prices affect their wealth, and some agents may be ruined in one state of environment and survive in some other state, even though the fundamentals of the economy are the same in all states. To study the indirect, or the adverse term-of-trade effect of extrinsic uncertainty on survival we need a dynamic economy.

Consider a discrete time, infinite horizon OLG economy with constant population. We use Gale's terminology [15] wherever appropriate. For expository simplicity, and without loss of generality we assume that at the beginning of every time period $t = 1, 2, \cdots$ there are two agents: one "young" born in t, and one "old" born in $t-1$. In period $t = 1$ there is one old agent of generation 0. There is one (perishable) consumption good in every period. The agent born in t (generation t) receives an endowment vector $e_t = (e_t^y, e_t^o)$ and consumes a vector $c_t = (c_t^y, c_t^o)$. We consider the Samuelson case [4] and assume, without loss of generality, $e_t = (1, 0)$. We assume that the preferences of the agent of generation t can be represented by expected utility function $\mathcal{U}_t(\cdot) = E\left[U^t(c_t)\right]$ with Bernoulli utility $U^t(c_t)$, continuously differentiable and almost everywhere twice continuously differentiable, strictly concave and strictly monotone in $\mathcal{D}$, compact, convex subset of R^2_{++}. The old agent of generation 0 is endowed with one unit of fiat money, the only nominal asset in the economy. In every period the market for the perishable consumption good is open and accessible to all agents. Denote the nominal price of the consumption good at time t by p_t. Define a *price system* to be a sequence of positive numbers, $\mathbf{p} = \{p_t\}_{t=0}^{\infty}$, a *consumption program* to be a sequence of pairs of positive numbers $\mathbf{c} = \{c_t\}_{t=0}^{\infty}$, a *feasible program* to be a consumption program that satisfies $c_t^y + c_{t-1}^o \le e_t^y + e_{t-1}^o = 1$. The agent of generation t

[4] If a population grows geometrically at the rate γ, so that γ^t agents is born in period t, and there is only one good in each period, the Samuelson case corresponds to marginal rate of intertemporal substitution of consumption under autarky, $U_1(e^y, e^o)/U_2(e^y, e^o)$, being less than γ. In our case $\gamma = 1$.

maximizes his lifetime expected utility in the beginning of period t. In period 1, the young agent gives its saving (s_1^y) of the consumption good, to the old agent in exchange for one unit of money (the exchange rate is determined by p_1). Thus, $p_1 s_1 = 1$. This unit of money is carried into period 2 (the old age of agent born in period 1) and is exchanged (at the rate determined by p_2) for the consumption food saved by the young agent born in period 2 (s_2^y). The process is repeated.

3.1 Perfect Foresight Equilibrium

If there is no uncertainty, with perfect foresight the price-taking young agent's optimization problem is the following:

$$\max U(c_t^y, c_t^o)$$

subject to

$$\begin{aligned} c_t^y &= 1 - s_t^y \\ c_t^o &= p_t s_t^y / p_{t+1} \end{aligned}$$

($0 \leq s_t^y \leq 1$, $t = 1, 2, \cdots$).

Here, $s_t^y \equiv e_t^y - c_t^y$ is savings of the young agent (this is the Samuelson case, in Gale's definitions [15]). A *perfect foresight competitive equilibrium* is defined as a feasible program and a price system such that

(i) the consumption program $\bar{\mathbf{c}} = \{\bar{\mathbf{c}}_\mathbf{t}\}$ solves optimization problem of each agent given $\bar{\mathbf{p}} = \{\bar{p}_t\}$: $(\bar{c}_t^y, \bar{c}_t^o) \in \mathcal{D}$, $\bar{c}_t^y = 1 - s_t$ and $\bar{c}_t^o = \bar{p}_t s_t / \bar{p}_{t+1}$ with

$$s_t = arg \max_{0 \leq s_t^y \leq 1} U\left((1 - s_t^y), s_t^y \frac{\bar{p}_t}{\bar{p}_{t+1}} \right)$$

and

(ii) the market for consumption good clears in every period:

$$\begin{aligned} \bar{c}_t^y + \bar{c}_{t-1}^o &= 1 \text{ (demand = supply for the consumption good)} \\ \bar{p}_t s_t &= 1 \text{ (demand = supply for money)} \end{aligned}$$

for $t = 1, 2, \cdots$.

By strict concavity of the utility function $U(c_t^y, c_t^o)$, the young agent's optimization problem has a unique solution. Hence, we can express s_t as a single-valued function of p_t/p_{t+1}, i.e. we write $s_t = s_t(p_t/p_{t+1})$. This function (called savings function) generates an offer curve in the space of net trades, as price ratios vary. In the perfect foresight equilibrium

$$s_t(p_t/p_{t+1}) = 1/p_t. \tag{20}$$

The stationary perfect foresight monetary equilibrium is a sequence of constant prices p and constant consumption programs $(1 - \bar{s}, \bar{s})$, where $\bar{s} = s(1)$.[5]

3.2 Sunspot Equilibrium

Now consider an extrinsic uncertainty in this economy. There is no uncertainty in fundamentals, such as endowments and preferences, but the agents believe that market prices depend on realization of an extrinsic random variable (sunspot). We assume that there is one-to-one mapping from the sunspot variable to price of the consumption good. Because the agents cannot observe future sunspots, they maximize expected utility over all possible future realization of the states of nature. We examine the situation with two states of nature, $\sigma \in \{\alpha, \beta\}$, that follow a first-order Markov process with stationary transition probabilities,

$$\Pi = \begin{bmatrix} \pi^{\alpha\alpha} & \pi^{\alpha\beta} \\ \pi^{\beta\alpha} & \pi^{\beta\beta} \end{bmatrix} \tag{21}$$

where $\pi^{\sigma\sigma'} > 0$ is the probability of being in state σ' in the next period given that current state is σ, and $\pi^{\sigma\alpha} + \pi^{\sigma\beta} = 1$. A young agent born in t observes price p_t^σ and solves the following optimization problem:

$$\max \left[\pi^{\sigma\alpha} U(c_t^{y,\sigma}, c_t^{o,\alpha}) + \pi^{\sigma\beta} U(c_t^{y,\sigma}, c_t^{o,\beta})\right]$$

subject to

$$\begin{aligned} c_t^{y,\sigma} &= 1 - s_t^\sigma \\ c_t^{o,\sigma'} &= p_t^\sigma s_t^\sigma / p_{t+1}^{\sigma'} \end{aligned}$$

$(0 \leq s_t^\sigma \leq 1,\ s_t^{\sigma'} \geq 0,\ \sigma, \sigma' \in \{\alpha, \beta\})$.

We restrict our attention to stationary equilibria, in which prices depend on the current realization of the state of nature σ, and do not depend on the calendar time nor the history of σ. A *stationary sunspot equilibrium*, SSE, is a pair of feasible programs and nominal prices, such that for every $\sigma \in \{\alpha, \beta\}$

(i) the consumption programs solve the agents' optimization problem:

$$\begin{aligned} &s^\sigma (p^\sigma / p^{\sigma'}) = \\ &arg \max_{0 \leq s^\sigma \leq 1} \left[\pi^{\sigma\alpha} U\left((1 - s^\sigma), s^\sigma p^\sigma / p^\alpha\right) + \right. \\ &\left. + \pi^{\sigma\beta} U\left((1 - s^\sigma), s^\sigma p^\sigma / p^\beta\right)\right] \end{aligned} \tag{22}$$

and

[5] Given our assumptions on preferences and endowments, the stationary perfect foresight monetary equilibrium exists and is optimal (see, for example, [21], Chap. 8).

(ii) markets clear in every period, in every state.

$$c^{y,\sigma} + c^{o,\sigma} = 1$$
$$p^{\sigma} s^{\sigma} = 1$$

It is easy to see that a stationary sunspot equilibrium exists when the equation

$$\frac{p^{\alpha}}{p^{\beta}} s^{\alpha} \left(\frac{p^{\alpha}}{p^{\beta}} \right) - s^{\beta} \left(\frac{p^{\beta}}{p^{\alpha}} \right) = 0 \tag{23}$$

has positive solutions for p^{α}/p^{β} other than 1 . Solution $p^{\alpha}/p^{\beta} = 1$ corresponds to the equilibrium in which uncertainty does not matter. It can be shown that, if sunspot equilibria exist in this economy, there are at least two of them, with $p^{\alpha}/p^{\beta} > 1$ and $p^{\alpha}/p^{\beta} < 1$ (see, for example, [7], [34]). This means that in the sunspot equilibrium consumption of old agents is above the certainty equilibrium consumption of olds in one state of nature and below in the other. Suppose, we introduce an exogenous minimal subsistence level of consumption (independent of $\sigma \in \{\alpha, \beta\}$). It may be the case that in one of the states of nature consumption of old agents falls short of minimal subsistence level: old agents are ruined. Note that the endowments are not affected by the uncertainty, and, therefore, there is no *direct* effect of uncertainty on ruin. The event of ruin is caused purely by an *indirect*, or term-of-trade effect: the equilibrium price system is such that the wealth of old agents does not allow them to survive. The following numerical example illustrates this possibility for the case of quadratic utility.

3.3 Ruin in Equilibrium

Let the preferences of the agents be represented by expected utility function with

$$U(\mathbf{c}) = u(c^y, c^o) - v(c^o)$$
$$u(c^y, c^o) = 2a\sqrt{c^y c^o} + q\, c^y + r\, c^o - \frac{1}{2} b (c^y)^2 - \frac{1}{2} d (c^o)^2$$
$$v(c^o) = \begin{cases} \frac{\theta}{2} \left(A - c^o\right)^2, & 0 < c^o \leq A \\ 0, & c^o > A \end{cases}$$

where a, b, c, q, r, θ, A are positive constants such that the utility function is increasing and jointly concave in its arguments in $\mathcal{D}$. $v(\cdot)$ is the disutility of consuming less than A, the minimal subsistence level.[6] As above, agents in

[6] It may seem odd that the disutility from starvation is finite, but this can be justified by the willingness of the agents to take a risk. Consider the following. In the continuous time, if the consumption of an old agent is above A, he lives to the end of the second period. If his consumption is below A, perhaps, he does not die immediately. Albeit low, the amount consumed allows him to live some time in

each generation receive identical positive endowments $e = 1$ of consumption good when young and zero endowments when old; the initial olds are endowed with one unit of money.

Benchmark case: perfect foresight

For the above preferences, savings function $s_t(p_t/p_{t+1})$ is implicitly defined by

$$\rho_t = \frac{a\sqrt{\rho_t s_t/(1-s_t)}) + q - b\,(1-s_t)}{a\sqrt{(1-s_t)/(\rho_t s_t)} + r - d\,\rho_t s_t - v'(\rho_t s_t)}, \tag{24}$$

where $\rho_t \equiv p_t/p_{t+1}$. The offer curve is described by

$$(1-x)\left(a\sqrt{\frac{y}{x}} + q - b\,x\right) - y\left(a\sqrt{\frac{x}{y}} + r - d\,y - v'(y)\right) = 0 \tag{25}$$

In the stationary (deterministic) perfect foresight monetary equilibrium consumption plan of an agent is $(x, 1-x)$, where x solves

$$a\left(\sqrt{\frac{x}{1-x}} - \sqrt{\frac{1-x}{x}}\right) + x\,(b+d) + v'(1-x) + q - r - b = 0 \tag{26}$$

Stationary sunspot equilibria

Two states of nature, α and β evolve according to a stationary first-order Markov process. The states of nature do not affect the endowments. Agents can trade their real and nominal assets. In a stationary sunspot equilibrium with trade s^α, s^β solve the following system of equations:

$$\begin{aligned} &\pi^{\alpha\alpha} a\sqrt{\frac{s^\alpha}{1-s^\alpha}} + (1-\pi^{\alpha\alpha})\,a\sqrt{\frac{s^\beta}{1-s^\alpha}} + q - b\,(1-s^\alpha) \\ &= \pi^{\alpha\alpha} a\left(\sqrt{\frac{1-s^\alpha}{s^\alpha}} + r - d\,s^\alpha - v'(s^\alpha)\right) \\ &\quad + (1-\pi^{\alpha\alpha})\left(a\sqrt{\frac{1-s^\alpha}{s^\beta}} + r - d\,s^\beta - v'(s^\beta)\right)\frac{s^\beta}{s^\alpha} \end{aligned} \tag{27}$$

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the second period, and his lifespan in the second period is the longer, the closer is his consumption to A. In the discrete time this translates into probability of survival in the second period as a function of consumption. Thus, the old agent survives with probability 1 if $c^o \geq A$ and with probability less than 1 if $c^o < A$. Suppose, the objective of the agent is to maximize the probability of survival (or maximize his expected lifespan). Then it can be presented equivalently as the objective to minimize the disutility from consumption at the level below A. Clearly, this disutility can be finite, at least in the vicinity of A, if the agent is willing to take a risk. The authors are indebted to David Easley for this argument.

and

$$\pi^{\beta\beta} a\sqrt{\frac{s^\beta}{1-s^\beta}} + \left(1-\pi^{\beta\beta}\right) a\sqrt{\frac{s^\alpha}{1-s^\beta}} + q - b\left(1-s^\beta\right)$$
$$= \pi^{\beta\beta} a\left(\sqrt{\frac{1-s^\beta}{s^\beta}} + r - d\, s^\beta - v'(s^\beta)\right) \tag{28}$$
$$+ \left(1-\pi^{\beta\beta}\right)\left(a\sqrt{\frac{1-s^\beta}{s^\alpha}} + r - d\, s^\alpha - v'(s^\alpha)\right)\frac{s^\alpha}{s^\beta}$$

It is easy to see that one solution is $s^\alpha = s^\beta = 1 - x$, where x solves the equation for the perfect foresight above. This solution does not depend on the transition probabilities, prices and consumption are not affected by the uncertainty: sunspots *do not matter* in this equilibrium. However, there may be more solutions. For example, for $a = 2$, $b = 0.5$, $d = 7$, $q = 0.02$, $r = 0.6\, \theta = 0.05\, A = 0.3$ and $\pi^{\alpha\alpha} = \pi^{\beta\beta} = 0.15$ there are three stationary monetary equilibria in the economy: one coinciding with the perfect foresight equilibrium and two sunspot equilibria. Prices and consumption programs for these equilibria are given in the following table.

Multiple Equilibria Under Extrinsic Uncertainty. *In every entry, the first number is consumption of the young, the second is consumption of the old, and the third is the nominal price of consumption good.*

State	PFE	1st SSE	2nd SSE
α	(0.6670; 0.3330; 3.00)	(0.5973; 0.4027; 2.48)	(0.7518; 0.2482; 4.03)
β	(0.6670; 0.3330; 3.00)	(0.7518; 0.2482; 4.03)	(0.5973; 0.4027; 2.48)

The consumption programs in sunspot equilibria are Pareto inferior to the program in the perfect foresight equilibrium. Furthermore, in two sunspot equilibria old agents survive in one state of nature and fail to survive in another *with the same amount of resources*, because equilibrium price is too high. (We intentionally considered the case where agents survive in the certainty equilibrium to demonstrate that survival is always feasible. Also, in this model young agents always survive, – otherwise, the overlapping generations structure collapses.)

4 Survival and Specialization

The entitlement approach in the study of famines suggests that one has to explicitly take into account: (1) some goods or services produced in the economy are less essential for survival than others, and (2) some groups of agents are involved in production of these "less essential" goods and services and supply them to the market in exchange for "more essential" ones, and, therefore, are more vulnerable to starvation. The first observation can be formalized as the existence of asymmetry in preferences, and the second one – as the existence of specialization in output (or endowments) among agents.

A model of a static economy with asymmetric preferences and complete specialization was introduced by Desai in [10]. The idea is the following. In order to survive an agent needs to consume an essential good (or goods) at or above some minimum level. Only after attaining the minimum level of consumption of the essential good the agent can derive utility from consumption of other, non-essential goods. If some agents in the economy are not initially endowed with the essential good, they have to purchase the essential good in the market. We modify Desai's model to incorporate the probability of survival when the consumption of the essential good falls below the minimum subsistence level[7]. There are two goods, essential ("food", labeled x) and non-essential ("non-food", labeled y) in consumption, and two agents, a food producer (agent 1) and a non-food producer (agent 2). The food producer is endowed with $f > 0$ units of good x, and the non-food producer with $e > 0$ units of good y. The agents have identical preferences, and each needs a minimum quantity of food, f_i^*, to survive. The probability $\mathcal{P}$ of survival of agent i is the function of i's consumption of food:

$$\mathcal{P}\left[i \text{ survives}\right] = \begin{cases} 0, & x_i \leq \bar{x}_i \\ v(x_i), & \bar{x} < x_i \leq f_i^* \\ 1, & x_i > f_i^* \end{cases}$$

where $v(x)$ is continuous and strictly increasing [8]. Without loss of generality we further assume $\bar{x}_i = 0$ and $f_i^* = f^*$ for $i = 1, 2$. Hence, if $x_i \in (0, f^*)$, i starves, and is ruined with probability $1 - v(x_i)$ because of starvation. Given his budget constraint, an agent i, first, maximizes his probability of survival (or minimizes probability of ruin), and, second, if he survives with probability one, maximizes his utility of consumption *above* the survival level:

$$u_i = u_i\left(\left(x_1 - f_i^*\right), x_2\right)$$

[7] This formulation eliminates the "degeneracy" of the equilibrium with starvation, in which an agent enjoys "minus infinity" utility, and, furthermore, prefers one "minus infinity" level of utility to another "minus infinity", see [10], p.434.

[8] More generally, probability function is non-decreasing and right-continuous. We use stronger assumptions to ensure uniqueness.

(the survival level of the non-essential good y is zero), where $u_i(\cdot)$ is strictly concave, strictly increasing and twice continuously differentiable. We can formally combine these two consecutive objectives of an agent into one objective function:

$$U_i\left(x_i, y_i\right) = \begin{cases} v\left(x_i\right), & x_i \leq f^* \\ v\left(f^*\right) + u\left(\left(x_i - f^*\right), y_i\right), & x_i > f^* \end{cases}$$

We assume

$$u(\cdot) = \left(x_i - f^*\right)^{\alpha_i} y_i^{\beta_i}, \tag{29}$$

$x_i > f^*$, $y_i > 0$, $\alpha_i, \beta_i \in (0, 1)$, $\alpha_i + \beta_i \leq 1$. The functional form of $v(x_i)$ in a static model is irrelevant, as long as it is strictly increasing in x_i. Let p be the price of food in terms of non-food consumption good. The *market equilibrium* in this economy is the set of consumption vectors $\left\{\left(x_1, y_1\right), \left(x_2, y_2\right)\right\}$ and price p such that

(i) given p, agent i maximizes his objective function U_i subject to his budget constraint:
$$p\, x_i + y_i \leq p\, f_i + e_i$$
$i = 1, 2$, $(f_1, e_1) = (f, 0)$, and $(f_2, e_2) = (0, e)$.

(ii) markets clear:
$$x_1 + x_2 = f,$$
$$y_1 + y_2 = e.$$

Because of the asymmetric preferences and complete specialization the non-food producer is more vulnerable to starvation [9]. Consider different cases.

Case 1. Absolute scarcity.
Suppose, the harvest is so low that it cannot feed the food producer himself: $f \leq f^*$. The food producer consumes all his endowment and survives with positive probability. There is no trade, and the non-food producer is ruined with probability 1.

Case 2. Aggregate scarcity.
If the endowment of the food producer exceeds his minimum subsistence level, he sells some of good x to the non-food producer in exchange for good y. However, if the total amount of food is not enough to feed both agents, $f^* < f < 2f^*$, the non-food producer starves and is ruined with positive probability.

Case 3. Aggregate availability.
Suppose now, that the total amount of food is enough to feed both agents, $f > 2f^*$. However, as the analysis below shows, this condition is necessary, but not sufficient to allow the non-food producer to survive with probability one in the equilibrium. Below we derive the necessary and sufficient condition of survival of both agents with probability one. We show that this condition does not involve endowment and preferences of the non-food producer.

[9] We assume free disposal for good y.

Now we proceed to the formal analysis. Case 1 is irrelevant for our purpose, because there is no trade in that case. In two other cases consumption of agent 1 (food producer) is

$$(x_1, y_1) = \left(f^* + \frac{\alpha_1}{\alpha_1 + \beta_1}(f - f^*), \frac{\beta_1}{\alpha_1 + \beta_1} p (f - f^*)\right).$$

At price p the wealth, in terms of food, of agent 2 (non-food producer) is e/p. If this wealth is below f^*, he sells all his endowment in good y for good x, and his consumption is then

$$(x_2, y_2) = \left(\frac{e}{p}, 0\right)$$

From the market clearance condition the equilibrium price is

$$p^* = \frac{\alpha_1 + \beta_1}{\beta_1} \frac{e}{f - f^*}. \tag{30}$$

Hence, agent 2 starves and is ruined with probability $1 - v\left(\frac{\beta_1}{\alpha_1+\beta_1}(f - f^*)\right)$ (notice, that this probability depends only on the endowment and preferences of agent 1). This happens when $e/p^* < f^* < f$, or, using (30),

$$f^* < f < f^*\left(2 + \frac{\alpha_1}{\beta_1}\right). \tag{31}$$

Clearly, it can happen that the non-food producer starves even when the amount of food is more than enough to feed both agents, i.e.,

$$2f^* < f < f^*\left(2 + \frac{\alpha_1}{\beta_1}\right).$$

This is an example of *exchange entitlement failure.*

If the wealth of agent 2 is above f^*, he survives with probability one, and his consumption is

$$(x_2, y_2) = \left(f^* + \frac{\alpha_2}{\alpha_2 + \beta_2}\left(\frac{e}{p} - f^*\right), \frac{\beta_2}{\alpha_2 + \beta_2}(e - pf^*)\right).$$

Equilibrium price in this case is

$$p^* = \frac{\alpha_2/(\alpha_2 + \beta_2)}{\beta_1/(\alpha_1 + \beta_1)} \frac{e}{f - f^*\left(1 + \frac{\beta_2(\alpha_1+\beta_1)}{\beta_1(\alpha_2+\beta_2)}\right)}. \tag{32}$$

Hence, the non-food producer survives with probability 1 when $e \geq p^* f^*$, or, using (32),

$$f \geq f^*\left(2 + \frac{\alpha_1}{\beta_1}\right). \tag{33}$$

This is the necessary and sufficient condition for survival of both types. One can see that, holding the aggregate amount of food fixed, agent 2 is more likely to starve, the more agent 1 prefers his own good over the other good. Neither non-food producer's tastes nor his endowment affect his probability of survival.

An important policy implication is that under condition (31) increase in the endowment of the non-food producer will not improve his food purchasing power. Hence, the famine remedy in this case can be (i) redistribution or (ii) direct food support. An interesting question in this regard is whether universal or targeted food support is more efficient. Drèze and Sen provide an extensive discussion of this issue in [12], Chapter 7. In the context of this model, the universal support means giving equal amount of food to both agents, and targeted support means giving the total amount of food aid to the most vulnerable agent, i.e. to the non-food producer. If the objective of the relief agency is to maximize the probability of survival of all agents, then in situation (31), because type 1 survives with probability 1 given his endowment, the relief agency can do either of the following:

(a) Give food to type 2 only. Type 2 will survive with probability 1 if the amount of food aid is, at least, $f_a \equiv f^* - \dfrac{\beta_1}{\alpha_1 + \beta_1}(f - f^*)$;

(b) Give equal amounts of food to both types. If each type receives $1/2 f_b$, such that $1/2 f_b < f^*$, then type 2 will survive with probability 1 if the value (in terms of food) of his endowment is at least $f^* - 1/2 f_b$. Simple calculations render $f_b = 2\left(f^* - \dfrac{f}{2 + \alpha_1/\beta_1}\right)$.

It is straightforward to show that $f_a < f_b$, i.e. targeted support is more efficient than universal support (requires less resources, holding cost of distribution equal), if and only if $f^* < f < f^*(1 + \alpha_1/2\beta_1)$. In other words, the model suggests that at relatively high aggregate amount of food in the economy the food aid from outside should be distributed equally among food- and non-food producers. At relatively low aggregate amount of food in the economy the food aid from outside should be directed to the non-food producers.

5 Concluding Remarks

Joan Robinson ([30], p.189) wrote that "the hidden hand will always do its work, but it may work by strangulation." It has of course been an achievement of high order to spell out the conditions under which the price system can play an effective role in coordinating decentralized decisions in order to generate a Pareto efficient allocation of resources. From all indications it appears, sadly enough, that the strangulation by the invisible hand will haunt millions, especially in the century of globalization. General equilibrium analysis is very much relevant in understanding the full implications of economic policy to improve the probability of survival. But help will not come from models in which uncertainty has no essential role to play or in which the consumers have the luxury of "choosing their life spans" explicitly or implicitly. Much remains to be done in developing models that can throw light on the survival issues.

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On Price-Taking Behavior in Asymmetric Information Economies*

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Summary. It is understood that rational expectations equilibria may not be incentive compatible: agents with private information may be able to affect prices through the information conveyed by their market behavior. We present a simple general equilibrium model to illustrate the connection between the notion of informational size presented in McLean and Postlewaite (2002) and the incentive properties of market equilibria. Specifically, we show that fully revealing market equilibria are not incentive compatible for an economy with few privately informed producers because of the producers' informational size, but that replicating the economy decreases agents' informational size. For sufficiently large economies, there exists an incentive compatible fully revealing market equilibrium.

1 Introduction

In many markets of interest, agents are asymmetrically informed. Sellers of stock or automobiles often possess information that potential buyers do not have. In the presence of informational asymmetries, prices may reveal information to some agents. A particularly low price for shares in a company may signal to an uninformed agent that better informed agents are not buying the stock, or may be selling the stock. The notion of rational expectations equilibrium is one generally accepted extension of Walrasian equilibrium to economies with asymmetrically informed agents. As in the case of symmetric information, agents are assumed to maximize expected utility in a rational expectations equilibrium. In the rational expectations model, however, agents maximize expected utility not with respect to an exogenously given probability distribution. Instead, agents maximize expected utility with respect to an updated probability distribution that combines their initial information with the additional information conveyed by the prices.

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Informational asymmetries can have serious consequences for the performance of an economy. While Walrasian equilibria are Pareto efficient under quite general conditions when agents are symmetrically informed, market outcomes in the simplest of economies can be inefficient when the agents are asymmetrically informed, as shown clearly in Akerlof's Lemon's paper (Akerlof (1970)). In at least one case, however, asymmetric information does not result in inefficiency. This is the case in which the rational expectations equilibrium is fully revealing, that is, when the price reflects all of the agents' information. If the price conveys all the information that agents have, then each agent's decision problem is equivalent to the problem that would be solved if all the information were publicly available. In this case, the welfare theorems assure that efficient outcomes are obtained.

Fully revealing rational expectations equilibria are ex-post efficient, but open the question of the reasonableness of the price-taking assumption. As in the case of Walrasian analysis of symmetric information economies, rational expectations equilibria assume that agents ignore the effect of their market behavior on prices. In economies with symmetrically informed agents, this assumption is sometimes justified by a heuristic argument that in large economies agents will not be able to affect the price. There is also a formal foundation for this argument that relies on the explicit modeling of agents' strategic possibilities in a general equilibrium setting and provides conditions under which the Nash equilibria of the strategic market game are approximately Walrasian.[4] Roughly, it can be shown that, in plausible strategic market games, agents will have little effect on the price of a good when they control a small portion of the good. This provides some justification that price-taking behavior is a plausible assumption for agents in large economies.

The situation with asymmetric information is more complicated since agents can affect the price not only through the quantity of a good that they trade, but also through the information their trades reveal. This second channel through which agents can affect prices means that it is not enough that the quantity of a good that an agent controls is small relative to the aggregate quantity of that good. An agent with a small amount of a particular good may affect the price of that good because of the information he possesses. It is well understood that there may be a conflict between the information contained in rational expectations equilibrium prices and an agent's incentive to reveal, directly or indirectly, his information.[5]

This conflict should not in itself be surprising, since the incentive *not* to take prices as given exists even when agents are symmetrically informed. The most that one would hope for is that the effect of an agent's behavior on prices, via the information that his market behavior reveals, will be negligible in large economies. Palfrey and Srivastava (1986) considered a stochastic replication

[4] See, e.g., Mas Colell, Dubey and Shubik (1980) or Postlewaite and Schmeidler (1978, 1981).

[5] See, e.g., Blume and Easley (1983).

procedure for an economy in which the incentive compatibility problems associated with rational expectations equilibria asymptotically vanish. However, their stochastic replication procedure has the property that, with probability one, each agent's private information is duplicated as the number of agents increases. In a large economy, a single agent's information is redundant in the presence of the information of all other agents.

We are interested in situations in which a single agent's information is *not* redundant. The prototypical large economy that we envision is one in which preferences and technology depend on the state of the world, which is not directly known. Each agent has some information (a signal) regarding the relative likelihoods of states. When agents' signals are conditionally independent (given the true state), the signal of a single agent can still provide additional information about the true state, even in the presence of many agents. However, the incremental value of that signal vanishes as the number of agents becomes large. In this world, agents become "informationally small" as economies grows, but they never become "informationally irrelevant." There is a large literature analyzing competitive models that ignores the asymmetric information that must surely be present in any real-world problem. The usefulness of analyses that ignore such asymmetric information hinges on the belief that the incentive problems brought on by asymmetrically informed agents become negligible in large economies.

We present and analyze a simple general equilibrium example with asymmetrically informed agents similar to that described above. In the example, asymmetrically informed agents make production decisions based on their private information. Markets then open in which the produced goods are traded. When the number of producers is small, the fully revealing market equilibria are not incentive compatible; an agent's market behavior can reveal private information, and the revealed information can affect prices in ways detrimental to that agent. Consequently, when agents take into account the informational impact of their market behavior, the outcome may be different from the competitive outcome. However, when the economy is replicated in a natural way, agents become informationally small, where the notion of informational size is essentially that introduced in McLean and Postlewaite (2002). As a consequence of their asymptotically vanishing informational size, agents will have no incentive to manipulate prices in large economies.

We discuss within the example several of the issues that arise in modelling general equilibrium economies with asymmetric information, including the completeness of markets and multiple equilibria.

2 Example

The economy. There are two states of nature that are equally likely, θ_1 and θ_2 and two periods. There are two kinds of agents, producers and consumers, and three commodities: type 1 widgets, type 2 widgets, and money (denoted m).

Producers. There are $\overline{n}$ producers, each of whom can make exactly one widget using his own labor and chooses which type to produce in the first period. The producers' choice of widgets is simultaneous. Producers have identical state independent payoff functions defined simply as their final holdings of money. That is, they value neither widgets nor their own labor input.

Consumers. Each consumer is endowed with 20 units of money but no widgets. Consumers have the same utility function $u(\cdot)$ that depends on the state, the number x_1 of type 1 widgets consumed, the number x_2 of type 2 widgets consumed and the final holding of money as given in the table below:

$$u(x_1, x_2, m, \theta_1) = m + 25x_1$$
$$u(x_1, x_2, m, \theta_2) = m + 10x_2$$

Note that, in state θ_i,only type i widgets yield positive utility.

Information. Prior to production, each producer receives a noisy signal of the state. The conditional distributions of the signal a producer receives in each state are given in the table below:

	state θ_1	θ_2
signal		
s_1	.8	.2
s_2	.2	.8

Producers' signals are conditionally independent.

Markets. There are no markets open in the first period. In the second period competitive markets open in which the widgets that have been produced can be exchanged for money. Since producers incur no opportunity cost in making widgets, each of the $\bar{n}$ producers makes a widget. We denote by n the number of widgets of type 1 produced in period 1 (hence, there are $\bar{n} - n$ type 2 widgets produced). If producers choose to make different types of widgets when they have observed s_1 than they make when they have observed s_2, the mix of widgets on the market in period 2 will convey information about the state θ. If consumers rationally take this information into account, the competitive price in period 2 will depend on the mix of widgets offered for exchange.

A strategy for a producer is a mapping $\sigma : \{s_1, s_2\} \to \{1, 2\}$ specifying which type of widget to produce as a function of the observed signal. We consider symmetric equilibria in which all producers employ the same (pure) strategy. If $\sigma(s_1) = \sigma(s_2)$, then no information will be conveyed by the mix of widgets on the market. However, if $\sigma(s_1) \neq \sigma(s_2)$, the number of widgets of type 1 will reveal the number of producers who received each of the two signals.

Consumers form expectations given the (common) strategy of the producers and the number of widgets of each type that are offered on the market in the second period. When $\sigma(s_1) \neq \sigma(s_2)$, Bayes rule uniquely determines the posterior distribution on Θ, but if $\sigma(s_1) = \sigma(s_2)$ there are producer choices lying off the equilibrium path where one or more producers produce the widget which was not the strategic choice for either of the signals. We denote by $\mu(\cdot|n) = (\mu(\theta_1|n), \mu(\theta_2|n))$, a consumer's beliefs when he observes n widgets of type 1 on the market. We assume that $\mu(\theta_1|n)$ is the Bayesian posterior probability on θ_1 when the number of widgets of type 1, n, is consistent with producers' (common) strategy σ, and unrestricted otherwise.[6] No restriction is placed on consumers' beliefs when n is not consistent with σ.

Let $J_{\bar{n}} = \{1, .., \bar{n}\}$. A price is a function $J_{\overline{n}} \to \mathbb{R}^2_+$ where $p(n) = (p_1^n, p_2^n)$ is the pair of prices for widgets 1 and 2 respectively when n widgets of type 1 are produced in period 1. (The price of m is normalized to 1.) Given a price function $p(\cdot)$, consumers maximize expected utility (with respect to their beliefs $\mu(\cdot|n)$). A market equilibrium is a price $p(\cdot)$ and optimizing behavioral rules for producers and consumers for which markets clear.[7]

Definition: Given beliefs, $n \mapsto \mu(\cdot|n)$, a *market equilibrium (ME)* is a price function $p(\cdot)$ and a common producer strategy $\sigma(\cdot)$, for which

i. The symmetric strategy profile $(\sigma(\cdot), ..., \sigma(\cdot))$ is a Bayes equilibrium and

ii. For each n, consumer demand for widgets at price $p(n)$ is equal to number the of widgets produced.

When consumers' have large initial endowments of money, they will want to purchase a large number of widgets of a given type if the expected utility of that type of widget is greater than the price. Similarly, they will purchase none if the expected utility is below the price. Consequently, the only possible market clearing prices for sufficiently large m are those for which the market price of each widget is equal to the expected value of that widget. The expected value of widget 1 when there are n widget 1's on the market is (given producer strategy σ) $25\mu(\theta_1|n)$, which then must be the price of widget 1 when n widget 1's are offered on the market. Analogously, the expected value and the price of widget 2 when n widget 1's are offered is $10\mu(\theta_2|n)$.

We are interested in the existence (or nonexistence) of a *fully revealing market equilibrium* (FRME), that is a market equilibrium for which the equilibrium price reveals the private information that agents (producers) have. When $\sigma(s_1) \neq \sigma_2(s_2)$, the price reflects the number of widgets of type 1 on the market, which is the same as the number of producers who have observed

[6] It will be consistent with σ if either (i) $\sigma(s_1) \neq \sigma(s_2)$; or (ii) $\sigma(s_1) = \sigma(s_2) = 1$ and $n = \bar{n}$; or (iii) $\sigma(s_1) = \sigma(s_2) = 2$ and $n = 0$.

[7] We use the term *market equilibrium* rather than *rational expectations equilibrium* because producers' choices may not maximize expected profit at the given prices since their decisions must be taken prior to time at which the market opens. We discuss this further in the last section.

signal s_1. Hence, a ME is fully revealing if the common producer strategy is separating. We will show that fully revealing ME cannot exist when the number of producers is too small, but they will exist if the number of producers is sufficiently large.

The case of a single producer

In the presence of a single producer, the problem reduces to the existence of a separating equilibrium in a simple sender-receiver game. In a separating equilibrium, the producer chooses one type of widget when he observes signal s_1 and the other type when he observes signal s_2.

Suppose that $\sigma(s_1) = 1$ and $\sigma(s_2) = 2$. Note first that, given this strategy, the consumers' beliefs are $\mu(\theta_1|1) = .8$ if he sees widget 1 and $\mu(\theta_1|0) = .2$ if he sees widget 2. The equilibrium prices must therefore be $p_1^1 = 25\mu(\theta_1|1) = 20$, $p_2^1 = 10\mu(\theta_2|1) = 2$ and $p_1^0 = 25\mu(\theta_1|0) = 5$, $p_2^0 = 10\mu(\theta_2|0) = 8$. We claim that this proposed strategy is not an equilibrium. To see this, suppose the producer receives signal s_2. If the producer produces widget 2, his payoff will be $p_2^0 = 8$, while his payoff from deviating and producing widget 1 is $p_1^1 = 20$. Thus, there cannot be a separating equilibrium in which the producer chooses widget 2 when he receives signal s_2.

The same type of argument demonstrates that there cannot be a separating equilibrium in which the producer chooses widget 2 when he receives signal 1. Thus there cannot be a fully revealing equilibrium when there is a single producer. We note that welfare is maximized when the producer chooses widget 1 after receiving signal 1 and widget 2 after signal 2.

The case of two producers

Suppose there are two producers. We will show that as in the case of a single producer, there can be no separating equilibrium. Let $\sigma(\cdot)$ be the common strategy and suppose that $\sigma(s_1) \neq \sigma(s_2)$. In particular, suppose that $\sigma(s_1) = 1$ and $\sigma(s_2) = 2$. Finally, suppose that producer 1 receives signal s_2. If both producers are following this strategy the consumers' beliefs about state 1 following 0, 1 or 2 widget 1's being offered on the market are given in the table below.

Widget Production	Consumers' beliefs	Expected value of widget 1	Expected value of widget 2
2 widget $1'$s	$\mu(\theta_1\|2) = .94$	23.5	.5
1 widget 1	$\mu(\theta_1\|1) = .5$	12.5	5
0 widget $1'$s	$\mu(\theta_1\|0) = .06$	1.5	9.4

As before, the market clearing price of widgets must be the expected value of the widget. If a producer chooses to produce a type 2 widget, then the most that he will get for this widget is 9.4. If he produces a type 1 widget, then his payoff is at least 12.5. Consequently it cannot be an equilibrium for producers to produce widget 1 following signal 1 and widget 2 following signal 2.

As in the case of a single producer, the calculations for the strategy which prescribes producing widget 1 following signal 2 and widget 2 following signal 1 are the same as in case 1. Consequently, this producer strategy will not be an equilibrium either.

In summary, when there are 2 producers there is no symmetric separating equilibrium. It is easy to see why: a consumer values widget 1 more highly in state θ_1 than he values widget 2 in θ_2. Producer 1 affects consumers' beliefs through his choice of widget. If consumers expect producers to choose widget 2 after seeing widget 2, a producer's payoff will be higher if he instead produces widget 1.

The case of many producers

When there are many producers, there will exist separating equilibria in which each producer chooses widget i after receiving signal s_i, $i = 1, 2$.[8] Suppose producers receive conditionally independent, noisy signals of the state that are accurate with probability .8 (that is, $P(s_i|\theta_i) = .8$). If $\overline{n}$ is large and if all producers follow the separating strategy proposed above, then, with high probability, approximately 80% of the widgets offered for sale will be of type 1 when the state is θ_1 (so that $n \approx .8\overline{n}$) and approximately 80% of the widgets offered for sale will be of type 2 when the state is θ_2 (so that $n \approx .2\overline{n}$). This observation is simply an application of the law of large numbers. If $\overline{n}$ is large and if approximately 80% of the widgets are of type 1 (i.e., if $n \approx .8\overline{n}$), a simple calculation verifies that the consumers' beliefs will ascribe probability close to 1 to state θ_1 (i.e., $(\mu(\theta_1|n), \mu(\theta_2|n)) \approx (1, 0)$). If $\overline{n}$ is large and if approximately 80% of the widgets are of type 2 (i.e., if $n \approx .2\overline{n}$), the same calculation verifies that the consumers' beliefs will ascribe probability close to 1 to state θ_2 (i.e., $(\mu(\theta_1|n), \mu(\theta_2|n)) \approx (0, 1)$). If the true but unobserved state is θ_1, then, with high probability, the price vector $(p_1^n, p_2^n) \approx (p_1^{.8\overline{n}}, p_2^{.8\overline{n}}) \approx (25, 0)$ and if the true but unobserved state is θ_2, then, with high probability, the price vector $(p_1^n, p_2^n) \approx (p_1^{.2\overline{n}}, p_2^{.2\overline{n}}) \approx (0, 10)$. When $\overline{n}$ is large, the production decision of a single producer has only a small effect on the ratio $\frac{n}{\overline{n}}$. If $\overline{n}$ is large, it follows that, with probability close to 1, any single producer who changes production from one type of widget to the other will have only a small effect on consumers' beliefs, and hence on the prices of the widgets.

Suppose that $\overline{n}$ is large and that producers are employimg the separating strategy. Consider a producer who receives signal s_1. He believes that the price of a type 1 widget will be close to 25 with probability $P(\theta_1|s_1) = .8$ and close to 0 with probability $P(\theta_2|s_1) = .2$, yielding an expected price of 20

[8] In addition, there will exist equilibria which are pooling, and hence, are not fully revealing. This is discussed in the last section.

for type 1 widgets. On the other hand, he believes that the price of a type 2 widget will be close to 0 with probability $P(\theta_1|s_1) = .8$ and close to 10 with probability $P(\theta_2|s_1) = .2$, yielding an expected price of 2 for type 2 widgets. Consequently, a producer who observes signal s_1 will produce a type 1 widget.

A similar calculation will be made by a producer who receives signal s_2. He believes that the price of a type 1 widget will be close to 25 with probability $P(\theta_1|s_2) = .2$ and close to 0 with probability $P(\theta_2|s_1) = .8$, yielding an expected price of 5 for type 1 widgets. On the other hand, he believes that the price of a type 2 widget will be close to 0 with probability $P(\theta_1|s_2) = .2$ and close to 10 with probability $P(\theta_2|s_2) = .8$, yielding an expected price of 8 for type 2 widgets. Consequently, a producer who observes signal s_2 will produce a type 2 widget..

In summary, there will exist a fully revealing market equilibrium when the number of agents is sufficiently large. It will be an equilibrium for each producer to produce the widget that maximizes expected value given his own information alone if other producers are doing the same. Any deviation from this will have a vanishingly small effect on price as the number of producers becomes large, making such deviations unprofitable.

3 Modelling the Consumer

The concept, market equilibrium, models producers as strategic, specifying precisely what actions are available to them, but does not do the same for consumers. In this section, we model the second stage game as a Shapley-Shubik market game (Shubik (1973), Shapley (1974)), in which producers put goods and consumers put money on widget-1 and widget-2 trading posts, with the prices determined to clear the markets. We will consider limits, as the number of consumers approaches infinity, of symmetric perfect Bayesian equilibria in which all producers offer their widgets for sale. These equilibria will define beliefs, producer strategies, and prices that constitute a market equilibrium. We view this section as justifying the simpler model of section 2.

Wherever possible, we maintain the notation of section 2. The timing of the game is now as follows. At stage 1, producers observe their signal, either s_1 or s_2, and decide which widget to produce. We restrict attention to equilibria in which all producers adopt the same production strategy, σ, and supply their widget to the appropriate trading post. At stage 2, consumers observe the number of widgets of each type that were produced, where n denotes the number of type 1 widgets and $\overline{n} - n$ is the number of type 2 widgets. After observing n, consumers decide how much money to bid for type 1 widgets and type 2 widgets, at their respective trading posts.

Let h be the number of consumers, and let j index a particular consumer. For $j = 1, ..., h$, let $b_1^j(n)$ denote the amount of money that consumer j bids on the widget-1 trading post when the number of type-1 widgets produced is n, and let $b_2^j(n)$ denote the amount of money that consumer j bids on the widget-2 trading post when the number of type-1 widgets produced is n. A strategy for consumer j is a mapping, $\psi^j : J_{\overline{n}} \to \mathbb{R}_+^2$, such that $b_1^j(n) + b_2^j(n) \leq 20$ holds for all n.

The market clears according to the following allocation rule.

$$x_1^j(n) = \frac{n b_1^j(n)}{\sum_{j'=1}^{h} b_1^{j'}(n)}, \tag{1}$$

$$x_2^j(n) = \frac{(\overline{n} - n) b_2^j(n)}{\sum_{j'=1}^{h} b_2^{j'}(n)}, \text{ and} \tag{2}$$

$$m^j(n) = 20 - b_1^j(n) - b_2^j(n),$$

where $x_1^j(n)$ denotes consumer j's purchases of widget 1 when the number of type-1 widgets produced is n, $x_2^j(n)$ denotes consumer j's purchases of widget 2 when the number of type-1 widgets produced is n, and $m^j(n)$ denotes consumer j's money consumption when the number of type-1 widgets produced is n. The money received by a firm selling a particular widget is the price of that widget. These prices are given by

$$p_1^n = \frac{\sum_{j'=1}^{h} b_1^{j'}(n)}{n} \text{ and}$$

$$p_2^n = \frac{\sum_{j'=1}^{h} b_2^{j'}(n)}{(\overline{n} - n)}.$$

The allocation rule guarantees that all trade on a market takes place at the same price, which is the total amount of money bid divided by the total amount of widgets supplied. From (3.1) and (3.2), we see that the percentage of the widgets up for sale that consumer j purchases is equal to the percentage of the money that consumer j bids. If numerator and denominator are both zero in (3.1) or (3.2), then consumers do not receive any widgets. Therefore, we adopt the convention that $\frac{0}{0} = 0$ in (3.1) and (3.2). However, prices are a different story. If, say, there are no type-1 widgets produced and no money is bid for type-1 widgets, then the price of type-1 widgets is indeterminate. The resolution of this indeterminacy is irrelevant for the characterization of perfect Bayesian equilibrium, but could affect the comparison to market equilibrium. We will comment on this later.

We restrict attention to symmetric perfect Bayesian equilibria, in which all widgets produced are supplied to the market. To find an equilibrium, we find consumer j's best response to the common strategy played by all other consumers, $(b_1(n), b_2(n))$, after n type-1 widgets are produced. We then impose the condition that consumer j's best response is in fact $(b_1(n), b_2(n))$. Given beliefs, μ, the optimization problem for consumer j is to choose $(b_1^j(n), b_2^j(n))$ to solve

$$\max[20 - b_1^j(n) - b_2^j(n)] + \mu(\theta_1 \mid n)\frac{25nb_1^j(n)}{b_1^j(n) + (h-1)b_1(n)}$$
$$+\mu(\theta_2 \mid n)\frac{10(\overline{n} - n)b_2^j(n)}{b_2^j(n) + (h-1)b_2(n)}.$$

Computing the first order conditions, imposing $(b_1^j(n), b_2^j(n)) = (b_1(n), b_2(n))$, and simplifying, we have

$$b_1(n) = \frac{\mu(\theta_1 \mid n)25n(h-1)}{h^2}$$
$$b_2(n) = \frac{\mu(\theta_2 \mid n)10(\overline{n} - n)(h-1)}{h^2}.$$

Notice that the above equilibrium bids are uniquely determined, as long as we impose symmetry. Plugging the above bids into the formula for prices, we have

$$p_1^n = (\frac{h-1}{h})25\mu(\theta_1 \mid n) \text{ and} \quad (3)$$
$$p_2^n = (\frac{h-1}{h})10\mu(\theta_2 \mid n). \quad (4)$$

The prices in (3.3) and (3.4) are uniquely determined from the ratio of bids and offers, except for p_1^n when $n = 0$ holds, and p_2^n when $n = \overline{n}$ holds. In these cases, bids and offers are zero, but either n or $(\overline{n} - n)$ appear in both the numerator and denominator, and cancel each other. Thus, we will define the *prices associated with a symmetric PBE* by (3.3) and (3.4).[9]

Proposition. *Consider a sequence* $(\sigma^h, \psi^h, \mu^h)_h$ *where* $(\sigma^h, \psi^h, \mu^h)$ *is a symmetric PBE for the game with h consumers and consider the associated sequence of prices,* $(p_1^{n,h}, p_2^{n,h})_h$*. If* $(\sigma^{h'}, \psi^{h'}, \mu^{h'})_{h'}$ *is a convergent subsequence, then* $(\lim_{h' \to \infty}(p_1^{n,h'}, p_2^{n,h'}), \lim_{h' \to \infty} \sigma^{h'})$ *is a market equilibrium for beliefs* $\lim_{h' \to \infty} \mu^{h'}$.

Proof. From the definition of PBE, and because there are only four possible producer strategies, σ^h, there exists $\overline{h}, \overline{\sigma}$, and $\overline{\mu}$ such that $h' > \overline{h}$ implies:

(1) $\sigma^h = \overline{\sigma}$, and

(2) $\mu^h(\theta_1 \mid n) = \overline{\mu}(\theta_1 \mid n)$ and $\mu^h(\theta_2 \mid n) = \overline{\mu}(\theta_2 \mid n)$ for all n occurring with positive probability, given $\overline{\sigma}$. Thus, $\overline{\mu}$ is consistent with $\overline{\sigma}$, according to the criterion required for a PBE. This also implies that $\overline{\mu}$ is consistent with $\overline{\sigma}$, according to the criterion required for a market equilibrium.

[9] The prices given by (3.3) and (3.4), for the case of markets with zero supply and demand, would arise if we placed ε offers of widgets on each market, and let ε approach zero. See the discussion of virtual prices in, say, Dubey and Shubik (1978).

>From (3.3) and (3.4), we see that, for $h' > \overline{h}$, the incentives for producers to deviate are exactly the same in the PBE as they are in a market equilibrium. Sequential rationality of $\overline{\sigma}$ in the PBE implies $\{\overline{\sigma}\}$ satisfies part (i) of the definition of a market equilibrium, given beliefs $\overline{\mu}$. The limiting price function, $p(n) = (25\overline{\mu}(\theta_1 \mid n), 10\overline{\mu}(\theta_2 \mid n))$, satisfies part (ii) of the definition of a market equilibrium. $\square$

4 Discussion

Incomplete markets

In the example, market equilibrium cannot be fully revealing when the number of producers is small, but is fully revealing when the number of producers is sufficiently large. Because producers are informationally small in large economies, they cannot gain by attempting to manipulate prices. However, even for large economies, a fully revealing market equilibrium is not a rational expectations equilibrium. In a rational expectations equilibrium, producers can observe the prices of widget-1 and widget-2, infer the state of nature, and produce the widget corresponding to the correct state. In the example, the market equilibrium is fully revealing, but only after output has been produced. A producer will produce the wrong widget with probability .2, so a market equilibrium is not ex post efficient. Also, if we were to change the parameter for observing the correct signal from .8 to .6, then there is no fully revealing market equilibrium. Producers are informationally small, so there is no incentive to manipulate market prices, but producers receiving signal s_2 are better off gambling that their signal is wrong and producing widget 1.

Markets are incomplete in the example: there is no forward market in which producers can sell their planned output before producing it. As mentioned in the introduction, the structure of markets is crucial for the example. Suppose instead that the only market available operated in the first period, in which producers could offer widgets of either type for delivery in the second period. Whatever prices prevail in this forward market, all producers will wish to sell the same widget – the widget with the higher price. Thus, it is impossible that producers with different signals will behave differently. But when all producers behave identically regardless of their information, the price cannot reflect their information.[10] On the other hand, suppose that a securities market operated in the first period, on which producers could trade money, contingent on whether the number of widgets produced was greater than, or less than, $\frac{\overline{n}}{2}$. Now production could depend on the prices of securities, so that full revelation would lead to efficient production decisions. In future work, we will explore

[10] This is similar to the phenomenon in Grossman and Stiglitz (1980).

the conjecture that, in this more complete market structure, fully revealing market equilibria exist, and correspond to rational expectations equilibria.

One might imagine a non-tatonement process that reveals producers' information (for example, a bargaining process between buyers and sellers), with trade taking place only after revelation has taken place. Assuming such an unmodelled process is unsatisfactory, however. The point of the present exercise is to understand when agents' private information will be revealed when those agents are behaving strategically with respect to the revelation. Any interesting analysis addressing this issue must model the process by which agents' information is reflected in prices. In other words, it is necessary to specify exactly what actions agents can take and the mapping of their actions into prices and outcomes.[11]

Specifying that producers choose which widgets to produce, with prices and outcomes arising from competitive behavior subsequent to the choices, provides a precise and plausible mechanism by which informed agents' information is incorporated into prices. One can, of course, think of alternative mechanisms that link agents' actions and resulting outcomes, but the intuition in the example is quite general. Whatever the mechanism linking actions and prices, if strategic behavior is modelled by Bayes equilibria, the revelation principle applies. An agent's incentive to misreport his information will be limited by the degree to which his report affects the expected price. Said differently, those agents whose information is likely to have a trivial effect on price have little to gain from misreporting that information. For many natural mechanisms, when the gains from altering behavior to affect the price are small, equilibrium actions will be close to actions that are optimal ignoring the effect on price.

Multiple equilibria

We have demonstrated the existence of a fully revealing incentive compatible ME when the number of producers was sufficiently large. This does not mean that all incentive compatible ME's are fully revealing. The nonrevealing ME in which sellers produce widget 1 regardless of their signal, at a price of $(12.5, 10)$ remains an incentive compatible ME. This will be a perfect Bayes equilibrium if consumers' beliefs following the disequilibrium choice of widget 2 by a producer were that that producer had seen signal s_1 with probability .5. Even if consumers beliefs were such that they believed that a producer who made this disequilibrium choice had seen signal s_2, this would have a negligible effect on the subsequent price when there are many producers. As a result, the return to a producer who chose to produce widget 2 would be lower than producing widget 1.

Our point is not that a large number of agents *necessarily* leads to information revelation but only that a large number (and the consequent infor-

[11] See Dubey, Geanakoplos and Shubik (1987) for an early argument along these lines and the general treatment of the question in Jackson and Peck (1999).

mational smallness) makes the return to manipulation of prices through the information revealed vanish asymptotically.

Interim vs. ex post incentive compatibility

The revealing ME is incentive compatible because at the time the seller makes his decision about which widget to produce, a change will have a small effect on the price with high probability. This is because the law of large numbers implies that for “most” realizations of the sellers’ signals, the posterior on Θ given the signals puts probability close to 1 on the true state, and any single deviation in the sellers’ choice of widgets will have a small effect on the posterior. For some realizations, however, a single seller’s change in the widget produced can have a nonnegligible effect on the posterior. Suppose that there are 1001 sellers, and the vector of signals s is such that 500 sellers receive s_1 and 501 sellers receive s_2. $P(\theta_1|s) = .4$ in this case. Consider, however, the vector of signals s' in which one s_2 is changed to an s_1; $P(\theta_1|s') = .6$. In other words, a single seller’s change in the choice of the widget to produce causes a nontrivial change in the posterior distribution on Θ, given the change in inference resulting from the production change. The nontrivial change, of course, translates into a nontrivial change in the market price.

Regardless of the number of replicas, a single seller’s actions will have a nontrivial effect on market prices for some realizations of the other agents’ signals. However, when the number of sellers is large, the probability that the other sellers’ signals are such that any given seller will have a nontrivial effect on the price is small. Since the potential gains from any change in price are bounded, the *expected* price change resulting from a change in production will be small when there are many sellers.

The presence of many other sellers makes a given seller informationally small. Given the other sellers’ information, the given seller’s signal provides little additional information, and the posterior distribution on Θ is not likely to be very sensitive to his information, and hence not likely to be sensitive to his market behavior.

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Ambiguity aversion and the absence of indexed debt*

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Summary. Following the seminal works of Schmeidler (1989), Gilboa and Schmeidler (1989), roughly put, an agent's subjective beliefs are said to be *ambiguous* if the beliefs may not be represented by a unique probability distribution, in the standard Bayesian fashion, but instead by a set of probabilities. An *ambiguity averse* decision maker evaluates an act by the minimum expected value that may be associated with it.

In spite of wide and long-standing support among economists for indexation of loan contracts there has been relatively little use of indexation, except in situations of extremely high inflation. The object of this paper is to provide a (theoretical) explanation for this puzzling phenomenon based on the hypothesis that economic agents are *ambiguity averse*. The present paper considers a general equilibrium model based on (Magill and Quinzii, 1997), with ambiguity averse agents, where both nominal and indexed bond contracts are available for trade and all relevant prices are determined endogenously. We obtain conditions which prompt an *endogenous* cessation of trade in indexed bonds: i.e., conditions under which there is no trade in indexed bonds in *any* equilibrium; only nominal bonds are traded. We argue that the obtained conditions mirror the known stylized facts about trade in indexed financial contracts.

1 Introduction

In spite of wide and long-standing support among economists for indexation of loan contracts there has been relatively little use of indexation, except

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in situations of extremely high inflation. Indeed, except in cases where inflationary circumstances forced them to do so, few governments, and fewer private borrowers, have issued indexed bonds. People seem to have a preference for specifying their obligations and opportunities in nominal units. As (Shiller, 1997) remarks:

> That the public should generally want to denominate contracts in currency units–despite all the evidence that it is not wise to do so and despite the obvious examples from nominal contracts of redistributions caused by unexpected inflation–should be regarded as one of the great economic puzzles of all time.

The object of this paper is to provide a (theoretical) explanation for this puzzling phenomenon based on the hypothesis that economic agents are *ambiguity averse*. The analysis throws up testable hypotheses and insights on policy that are distinctive, compared to what a more standard analysis based on the assumption that decision makers are (subjective) expected utility maximizers would suggest.

Suppose an agent's subjective knowledge about the likelihood of contingent events is consistent with more than one probability distribution. And further that, what the agent knows does not inform him of a (second order) probability distribution over the set of 'possible' (first order) probabilities. Roughly put, we say then that the agent's beliefs about contingent events are characterized by *ambiguity.* If ambiguous, the agent's beliefs are captured not by a unique probability distribution in the standard Bayesian fashion but instead by a set of probabilities. Thus not only is the outcome of an act uncertain but *also* the expected payoff of the action, since the payoff may be measured with respect to more than one probability. An *ambiguity averse* decision maker evaluates an act by the minimum expected value that may be associated with it: the decision rule is to compute all possible expected values for each action and then choose the act which has the best minimum expected outcome. This notion of ambiguity aversion, an intuition about behavior under subjective uncertainly famously noted in (Ellsberg, 1961) and earlier by (Knight, 1921), inspires the formal model of Choquet expected utility (CEU) preferences introduced in (Schmeidler, 1989). The present paper considers a competitive general equilibrium model of goods, bonds and money markets populated by agents with CEU preferences[3], where both nominal and indexed bonds are available for trade and prices of all goods and bonds are determined

[3]Recent literature has debated the merits of the CEU framework as a model of ambiguity aversion. For instance, (Epstein, 1999) contends that CEU preferences associated with *convex capacities* (see section 2, below) do not always conform with a "natural" notion of ambiguity averse behavior. On the other hand, (Ghirardato and Marinacci, 2002) argue that ambiguity aversion is demonstrated in the CEU model by a broad class of capacities which includes convex capacities.

Further on it will be seen that a property of portfolio inertia typical to the behavior of CEU agents is what crucially underpins the whole argument in the paper.

endogenously. We obtain conditions which prompt an *endogenous* cessation of trade in indexed bonds among private agents: i.e., conditions under which there is no trade in indexed bonds in *any* equilibrium and only nominal bonds are traded. It is worth clarifying, at this point, that while we "explain" the veritable absence of indexed debt by showing that no trade in indexed bonds is the unique equilibrium outcome under certain conditions, the analysis does not imply that this is an efficient outcome. Indeed, as will be evident from the analysis, making appropriate changes to the way price indices are constructed would lead, if the theory presented in this paper is empirically valid, to more widespread indexation of debt *and* an accompanying Pareto improvement.

An important point of inspiration for the analysis was to note that indexing does not eliminate all (price) risk—rather it substitutes one risk for another—an observation, we believe, originally due to (Magill and Quinzii, 1997). An indexed bond, whose payoffs by definition are denoted in units of a reference bundle of goods and services, will be secure against the aggregate price level risk arising from changes in the money supply—*the monetary risk*, but unavoidably picks up the *real risks* arising from fluctuations in the relative prices of the goods in the reference bundle. A nominal contract on the other hand implies susceptibility to monetary risk but less so to real risk. The basic intuition is straightforward. Being paid in terms of an index essentially amounts to being paid units of the reference bundle of goods. Typically, the reference bundle contains items that are not part of a given individual's consumption basket. Hence, effectively the individual is left to exchange goods in the reference bundle not in his consumption basket with goods he actually consumes. Thus, a change in the price of goods not in his basket will affect the "worth" of his remuneration in terms of the goods he does consume. Since the presence of both types of risk is typical, standard portfolio analysis will advise that the optimal portfolio should contain *both* nominal and indexed contracts (or to put it somewhat differently, partial indexation). Given this one would expect, and a result in this paper confirms, trade in indexed bonds will always be observed in a market consisting of SEU (subjective expected utility) agents so long as there were *some* inflation, however small. Under ambiguity aversion the market outcome, though, may be dramatically different.

More specifically suppose, with respect to *any* two agents wishing to trade in indexed bonds, the following is true:

1. the indexation bundle contains at least one good which is not consumed by either of the agents;
2. the agents' beliefs about the change in the price of good(s) not consumed by either of them, relative to the average price level, is ambiguous;
3. agents are ambiguity averse.

Mukerji and Tallon (2003a) shows that this inertia property may be derived from the primitive notion of ambiguity presented in Epstein and Zhang (2001) without relying on a parametric preference model such as the CEU.

The main result of the paper shows that, if agents believe general inflation will not exceed a given bound and if ambiguity of beliefs about the relative price movements is sufficiently high, then agents will have zero holdings of the indexed bond in *any* equilibrium.

The result appears to fit well with what is commonly observed, both, in terms of the plausibility of the hypotheses it rests on and in terms of its consistency with regularities widely associated with trade in indexed debt. First, consider the plausibility of the assumptions. Even at the best of times and even in the most developed nations, information (say, formal forecasts) about *relative* price movements are very hard to come by. Pick any two agents in the economy; it is inevitable that the consumer price index will include goods and services that are not part of the consumption basket of either agent. For instance, it will inevitably include housing in regions that the agents have no interest in. It is a plausible assumption that agents will have, at best, very sparse informal knowledge about possible (relative) price movements of goods and services that they never consume. Thus, if agents are typically ambiguity averse when confronted with vague information, as much experimental evidence suggests (see (Camerer, 1995)), then it would seem compelling (a priori) to argue that they would behave in an ambiguity averse manner when acting on beliefs about relative price movements of goods that never figure in their consumption plans[4].

Next, consider some "stylized facts" about trade in indexed debt. As has been already mentioned, barring certain exceptions trade in indexed bonds, especially private bonds, is negligible. The exceptions are, one, situations of extremely high and variable inflation, and two, situations (like in U.K., Israel) where even though inflation is currently moderate, many wage payments are statutorily index linked. A second exception occurs in economies which have tamed inflation in the recent past but have experienced bouts of high inflation in the more distant past (many S. American countries and also, Israel). It will be explained that the logic of the main result does not only imply that trade in indexed debt will be observed during episodes of relatively high inflation; such an episode actually serves to ensure that trade in indexed debt would endure for many periods beyond the original trigger even if agents do not expect further bouts of high inflation. It is also observed that individuals in countries with high and variable inflation commonly denominate their debt (or even transactions like rental contracts for housing, as in Israel) in U.S. Dollars even though they may rarely choose to tie payments to an index like the CPI. This practice, at the least, demonstrates that agents understand the vulnerability of nominal contracts denominated in terms of the (relatively

[4]It is interesting to note in this context, as reported in (Shiller, 1997), when asked for reasons for not opting for indexation many agents say that they are inhibited by their doubts that the government inflation numbers were valid for their individual circumstances (pp 183, 188-190, 208). (Apparently, the concern was that the official price index referred to a basket of goods that was possibly different from the individual's.)

inflationary) domestic currency and *do act* on their understanding. They are not as “naive” or “confused” as may be supposed in the first instance. If anything, this observation deepens the puzzle: if they do understand the point of indexation why do they refuse to use CPI and use the Dollar instead? The results presented in the paper provide an explanation of this practice. Finally, even though ours is an analysis of the private debt market in that we do not model government behavior per se, we will argue the phenomenon actually modelled here also provides an explanation for the lack of trade in indexed public debt.

Finally, we relate our paper to some of the literature that has applied CEU preferences to financial asset choice and financial market equilibrium. Although a fuller discussion of the precise relationship between this contribution and existing work is best deferred till the model has been spelled out, some preliminary remarks might help the reader to grasp the intuition behind our result. CEU preferences were identified by (Dow and Werlang, 1992) as a potential source of inertia in portfolio holding: an agent having a *riskless* endowment will not want to hold an uncertain asset on a (degenerate) price interval only if he perceives the asset return as ambiguous and if he is ambiguity averse. As noted in (Mukerji and Tallon, 2001) this inertia result does not translate to no trade in an equilibrium model unless some extra ingredients are added. Following a result in (Epstein and Wang, 1994) on sufficient conditions for an equilibrium to be supported by multiple asset prices, we precisely identified in that paper what these ingredients were: part of the asset return has to be idiosyncratic (i.e., uncorrelated with the agents’ endowments), and that part has to be ambiguous. Then, we showed, it is possible to establish that assets that are of this form may not be traded at any equilibrium. The present paper may be seen to be following that line of research by specifying what form these idiosyncrasies might take in a more concrete setting: here, they are precisely the ”noise” introduced in the return of an indexed asset *via* the presence of ”irrelevant” goods (irrelevant in the sense that the agents trading the asset neither consume nor are endowed with them) in the indexation bundle. The way this paper ”operationalizes” the idea is by restricting the financial market trade to agents who consume a bundle of goods that does not include some of the goods that are used in the indexation bundle. The ”irrelevant” goods are consumed by some other ”prop” agents who do not participate in financial markets; their role in the model is to endogenize the (stochastic) prices of these goods. Their role is crucial in the sense that without them we would lose the source of the noise and therefore the no trade result.

The rest of the paper is organized as follows. The following section provides an introduction to the formal model of Choquet expected utility. Section 3 works through a leading example with the aim of conveying the essential intuition of the argument in a partial equilibrium setting. Section 4 contains the general equilibrium model and the main result. Section 5 concludes the paper with a discussion of the related literature as well as the interpretation and empirical significance of the findings.

2 Ambiguity aversion

2.1 The Ellsberg urn

One classic experiment illustrating how ambiguity aversion may affect *behavior*, due to Daniel Ellsberg (1961), runs as follows:

> There are two urns each containing one hundred balls. Each ball is either red or black. The subjects are told of the fact that there are fifty balls of each color in urn I. But no information is provided about the proportion of red and black balls in urn II. One ball is chosen at random from each urn. There are four events, denoted IR, IB, IIR, IIB, where IR denotes the event that the ball chosen from urn I is red, etc. On each of the events a bet is offered: \$100 if the events occurs and \$0 if it does not.

The modal response is for a subject to prefer every bet from urn I (IR or IB) to every bet from urn II (IIR or IIB). That is, the typical revealed preference is $IB \succ \ IIB$ *and* $IR \succ IIR$. (The preferences are strict.) Clearly, the decision maker's beliefs about the likelihood of the events, as revealed in the preferences, is not consistent with a unique probabilistic prior. The story goes: People dislike the ambiguity that comes with choice under uncertainty; they dislike the possibility that they may have the odds wrong and so make a wrong choice (*ex ante*). Hence they go with the gamble where they know the odds — betting from urn I. A slight restatement of the intuition conveyed by the observed (modal) choice provides a useful perspective on what is to follow. Notice, *betting on* IIR is the same as *betting against* IIB, and vice versa, since the events are complementary. But to decide whether to bet on or against IIR requires information about relative likelihood of the event IIR and its complement. That is, of course, what ambiguity about IIR precludes. On the other hand, betting on, say, IB allows the decision maker (DM) to choose a prospect whose evaluation is unaffected by ambiguity.

2.2 Choquet expected utility

Let $\Omega = \{\omega_i\}_{i=1}^{N}$ be a finite state space, and assume that the DM chooses among acts with state contingent payoffs, $z : \Omega \to \mathbb{R}$. In the CEU model ((Schmeidler, 1989)) an ambiguity averse DM's subjective belief is represented by a *convex non-additive probability* function (or a *convex capacity*), ν such that, (i) $\nu(\emptyset) = 0$, (ii) $\nu(\Omega) = 1$ and, (iii) $\nu(X \cup Y) \geq \nu(X) + \nu(Y) - \nu(X \cap Y)$, for all $X, Y \subset \Omega$. Define the *core* of ν, (notation: $\Delta(\Omega)$ is the set of all additive probability measures on Ω):

$$\mathcal{C}(\nu) = \{\pi \in \Delta(\Omega) \mid \pi(X) \geq \nu(X), \text{ for all } X \subseteq \Omega.\}$$

Hence, $\nu(X) = \min_{\pi \in \mathcal{C}(\nu)} \pi(X)$. Hence, convex capacity may be interpreted as representing a convex set of (additive) probabilities. The *ambiguity*[5] of the belief about an event X is measured by the expression $\mathcal{A}(X;\nu) \equiv 1 - \nu(X) - \nu(X^c) = \max_{\pi \in \mathcal{C}(\nu)} \pi(X) - \min_{\pi \in \mathcal{C}(\nu)} \pi(X)$.

Like in SEU, a *utility function* $u : \mathbb{R}_+ \to \mathbb{R}$, $u'(\cdot) \geq 0$, describes DM's attitude to risk and wealth. (Gilboa and Schmeidler, 1989) showed, that given a convex non-additive probability ν, *the Choquet expected utility*[6] of an act is simply the minimum of all possible 'standard' expected utility values obtained by measuring the contingent utilities possible from the act with respect to each of the additive probabilities in the core of ν:

$$\mathbb{CE}_\nu u(z) = \min_{\pi \in \mathcal{C}(\nu)} \left\{ \sum_{\omega \in \Omega} u(z(\omega))\, \pi(\omega) \right\} \equiv \int_\Omega u(z(\omega))\, d\nu$$

The fact that the same additive probability in $\mathcal{C}(\nu)$ will not in general 'minimize' the expectation for two different acts, explains why the Choquet expectations operator is not additive, i.e., given any acts $z, w : \mathbb{CE}_\nu(z) + \mathbb{CE}_\nu(w) \leq \mathbb{CE}_\nu(z+w)$. The operator is additive, however, if the two acts z and w are *comonotonic*, i.e., if $(z(\omega_i) - z(\omega_j))(w(\omega_i) - w(\omega_j)) \geq 0$.

Next, we state the notion of independence of convex non-additive probabilities, proposed by (Gilboa and Schmeidler, 1989), used in this paper. Essentially, the idea is as follows. Start with the set of probabilities in the core of each capacity, select a probability from each such set and multiply to obtain the corresponding product probability in the usual way and repeat for all possible selections, thereby obtaining a set of product probabilities. The lower envelope of the set of product probabilities, obtained in this way, is the product capacity.

Definition 1. *Let ν and μ be two convex non-additive probabilities, defined on contingency spaces Ω_ν and Ω_μ respectively. The independent product of ν and μ , denoted $\nu \bigotimes \mu$, is defined as follows*

$$\left(\nu \bigotimes \mu\right)(A) \equiv \min\left\{(\pi_\nu \times \pi_\mu)(A) : \pi_\nu \in \mathcal{C}(\nu), \pi_\mu \in \mathcal{C}(\mu)\right\}$$

for every $A \subseteq \Omega_\nu \times \Omega_\mu$.

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[5](Fishburn, 1993) provides an axiomatic framework for this definition of ambiguity and (Mukerji, 1997) demonstrates its equivalence to a more primitive and epistemic notion of ambiguity (expressed in term's of the DM's knowledge of the state space).

[6]The Choquet expectation operator may be directly defined with respect to a non-additive probability, see (Schmeidler, 1989). Also, for an intuitive introduction to the CEU model see Section 2 in (Mukerji, 1998).

It is well-known that it is possible to define more than one notion of independence for non-additive beliefs. (Ghirardato, 1997) presents a comprehensive analysis of the various notions. The definition invoked above, suggested by (Gilboa and Schmeidler, 1989), is arguably the more prominent in the literature. However, the formal analysis in the present paper, given the primitives of our model, does not hinge on this particular choice of the notion of independence. An important finding of Ghirardato's analysis was that the proposed specific notions of independent product give rise to a unique product for cases in which marginals have some additional structural properties. The capacity we use in our model is a product of two two-point capacities (i.e., each capacity is defined on a state-space consisting of two states). A two-point capacity is, of course, a convex capacity and (trivially) a belief function. As is explicit in Theorems 2 and 3 in (Ghirardato, 1997), if marginals satisfy the structural properties the marginals we use do, then uniqueness of product capacity obtains.

2.3 A portfolio inertia

(Dow and Werlang, 1992), identified an important implication of Schmeidler's model. They showed, in a model with one risky and one riskless asset, that if a CEU maximizer has a riskless endowment then there exists a *set* of asset prices that support the optimal choice of a riskless portfolio. The intuition behind this finding may be grasped in the following example. Consider an asset that pays off 1 in state L and 3 in state H and assume that $\nu(L) = 0.3$ and $\nu(H) = 0.4$. Assuming that the DM has a linear utility function, the expected payoff of buying an unit of z, the act z_b, is given by $\mathbb{CE}_\nu(z_b) = 0.6 \times 1 + 0.4 \times 3 = 1.8$. On the other hand, the payoff from going short on an unit of z (the act z_s) is higher at L than at H. Hence, the relevant minimizing probability when evaluating $\mathbb{CE}_\nu(z_b)$ is that probability in $\mathcal{C}(\nu)$ that puts most weight on H. Thus, $\mathbb{CE}_\nu(z_s) = 0.3 \times (-1) + 0.7 \times (-3) = -2.4$. Hence, if the price of the asset z were to lie in the open interval $(1.8, 2.4)$, then the investor would strictly prefer a zero position to either going short or buying. Unlike in the case of unambiguous beliefs there is no single price at which to switch from buying to selling. Taking a zero position on the risky asset has the unique advantage that its evaluation is not affected by ambiguity.

3 The single decision maker's problem: the intuition in a simplified set up

In this section we consider the problem of a decision maker (DM) who wants to transfer an amount S, $S > 0$, from today (Period 0) to tomorrow (Period 1). Goods prices that will obtain tomorrow are uncertain at the moment and, for the purposes of this section, taken to be exogenously determined. We will examine, in particular, the DM's choice of portfolio given that the DM has access to only two kinds of assets, nominal bonds and indexed bonds, whose prices are known and exogenously given. While the model in this section is simpler in many details than the one in the next section, it is instructive in that it will reveal to us how the trade-offs involved, given ambiguity aversion, are such that the DM will strictly prefer to maintain a zero holding of the indexed bond over a *non-degenerate* interval of indexed bond prices. This, as we will see in the next section, is a key intuition to understanding why no trade in indexed bonds might emerge as an equilibrium outcome.

3.1 A simple portfolio problem

We assume that there are just two goods in the economy: x and y. The agent consumes only good x and is endowed in Period 1 with a (non-random) endowment of that good, $\bar{x}$. The agent does not consume good y nor is he endowed with that good. However, the indexed bond pays off a unit of good x and a unit of good y. The nominal bond pays in units of money.

The money supply in the economy in Period 1 can be either high (M) or low (m). When the money supply is low, suppose that prices can be equal to either (p_x, p_y^L) or (p_x, p_y^H), with $p_y^H > p_y^L$, *i.e.*, we assume that the price of good y can be affected by factors that do not affect the price of good x. When the money supply is high, we assume that prices can be either equal to $(\lambda p_x, \lambda p_y^L)$ or $(\lambda p_x, \lambda p_y^H)$ where $\lambda = M/m > 1$. This is reminiscent of the quantity theory of money.

The following four states exhaustively describe the price uncertainty faced by the individual:

State	Prices	Return from an indexed bond
1	(p_x, p_y^H)	$p_x + p_y^H$
2	$(\lambda p_x, \lambda p_y^H)$	$\lambda \times (p_x + p_y^H)$
3	(p_x, p_y^L)	$p_x + p_y^L$
4	$(\lambda p_x, \lambda p_y^L)$	$\lambda \times (p_x + p_y^L)$

In this section we leave the decision problem concerning the Period 0 consumption unspecified and simply assume that the agent wants to save a given amount S. Let x^s denote the agent's consumption in state s, b^i the agent's indexed bond holding, q^i its price, b^n the agent's nominal bond holding, and q^n its price. The DM's budget constraints may then be rewritten to obtain:

$$
\begin{aligned}
x^1 &= \bar{x} + \left(1 + \frac{p_y^H}{p_x}\right) b^i + \frac{b^n}{p_x} &&= \bar{x} + \left(1 + \frac{p_y^H}{p_x} - \frac{q^i}{q^n p_x}\right) b^i + \frac{S}{q^n p_x} \\
x^2 &= \bar{x} + \left(1 + \frac{p_y^H}{p_x}\right) b^i + \frac{b^n}{\lambda p_x} &&= \bar{x} + \left(1 + \frac{p_y^H}{p_x} - \frac{q^i}{q^n \lambda p_x}\right) b^i + \frac{S}{q^n \lambda p_x} \\
x^3 &= \bar{x} + \left(1 + \frac{p_y^L}{p_x}\right) b^i + \frac{b^n}{p_x} &&= \bar{x} + \left(1 + \frac{p_y^L}{p_x} - \frac{q^i}{q^n p_x}\right) b^i + \frac{S}{q^n p_x} \\
x^4 &= \bar{x} + \left(1 + \frac{p_y^L}{p_x}\right) b^i + \frac{b^n}{\lambda p_x} &&= \bar{x} + \left(1 + \frac{p_y^L}{p_x} - \frac{q^i}{q^n \lambda p_x}\right) b^i + \frac{S}{q^n \lambda p_x}
\end{aligned}
$$

The budget constraints reveal how each of the two kinds of bonds provide a hedge against a particular type of risk while simultaneously making the agent vulnerable to another type of risk. The agent does not consume y, hence given that the indexed bond pays a unit each of x and y, on maturity (of the indexed bond) the agent is effectively left to exchange units of y obtained for units of x. Therefore, even though payoff from an indexed bond is immune to monetary shocks (it is independent of λ) it changes with changes in the price of y, relative to the price of x. On the other hand, while the payoff (to the agent) of a nominal bond is not affected by shocks to the relative price of y, it is affected by monetary shocks (i.e., the value of λ). Hence, if $b^i = 0$, $x^1 = x^3$ and $x^2 = x^4$, while, if $b^n = 0$, then $x^1 = x^2$ and $x^3 = x^4$. Notice also that, given our assumptions, if $b^i > 0$, then $x^1 > x^3$ and $x^2 > x^4$, while, if $b^i < 0$, then $x^1 < x^3$ and $x^2 < x^4$; i.e., the agent's ranking of the states (1,3 and 2,4) according to consumption *reverses* when switching from a long to a short position on the indexed bond.

We next explore the consequences of ambiguity of beliefs about relative price movements on the agent's decision whether or not hold indexed bonds. We assume that the agent is risk averse, with a utility index $u : \mathbb{R}_+ \to \mathbb{R}$, which is increasing, strictly concave and differentiable. Suppose the agent has precise probabilistic beliefs concerning the money supply[7] (and consequently whether the "price level" is high or low) but has ambiguous beliefs concerning the realization of the idiosyncratic shock that affects only the price of good y (a good he is not endowed with, and which he does not consume). More precisely, we assume that the agent can assess the probability of the event $\{1, 3\}$ to be, say, μ and that of event $\{2, 4\}$ to be $1 - \mu$. On the other hand, conditional on a monetary state, the agent has only vague beliefs on whether the price of good y is high or low, which is represented by the fact that subjective beliefs are described by capacities $\nu^H \equiv \nu(\{1, 2\})$ and $\nu^L \equiv \nu(\{3, 4\})$, with $\nu^L + \nu^H < 1$. We then assume that the overall beliefs of the agent are simply the independent product of μ and ν. The preferences of the agent are then represented by a utility functional, denoted $V(x^1, x^2, x^3, x^4)$, obtained by taking the Choquet integral of $u(x^s)$ with respect to the independent product belief $\mu \otimes \nu$.

If $b^i > 0$, then $V(x^1, x^2, x^3, x^4)$ is given by:

$$\mu\left(\nu^H u(x^1) + (1 - \nu^H)u(x^3)\right) + (1 - \mu)\left(\nu^H u(x^2) + (1 - \nu^H)u(x^4)\right)$$

If $b^i < 0$, then $V(x^1, x^2, x^3, x^4)$ is given by:

$$\mu\left((1 - \nu^L)u(x^1) + \nu^L u(x^3)\right) + (1 - \mu)\left((1 - \nu^L)u(x^2) + \nu^L u(x^4)\right)$$

Note that if $\nu^H + \nu^L = 1$ then the two expressions above coincide.

[7] The actual equilibrium model in the next section allows beliefs about the money supply to be ambiguous too.

3.2 A price interval supporting zero holding of the indexed bond

We now establish, following (Dow and Werlang, 1992) that there is a non-degenerate interval of relative bond prices, $\frac{q^i}{q^n}$, at which the agent optimally wants to hold a zero position in the indexed bond. Below, we present an informal, intuitive argument. A more formal argument appears in the Appendix.

Suppose the agent presently holds only nominal bonds and is considering buying/selling an arbitrarily small unit of indexed bonds. The agent's present utility level in each of the four states may then be represented generically by $U(\lambda) \equiv u\left(\bar{x} + \frac{S}{q^n \lambda p_x}\right)$, where $\lambda = 1$ in states 1 and 3 and $\lambda = 1+\varepsilon$, $\varepsilon > 0$, in states 2 and 4. Since $u''(x) < 0$, the marginal utility in each state, $U'(\lambda) \equiv u'\left(\bar{x} + \frac{S}{q^n \lambda p_x}\right)$, is increasing in λ; the intuition being that a higher inflation will affect the saver adversely. Now consider the gross increase (decrease) in welfare at each state if the agent were to buy (sell) an infinitesimal unit of an indexed bond:

$$\text{State 1}: \left(1 + \frac{p_y^H}{p_x}\right) U'(1)$$
$$\text{State 2}: \left(1 + \frac{p_y^H}{p_x}\right) U'(1+\varepsilon)$$
$$\text{State 3}: \left(1 + \frac{p_y^L}{p_x}\right) U'(1)$$
$$\text{State 4}: \left(1 + \frac{p_y^L}{p_x}\right) U'(1+\varepsilon)$$

Figure 1, depicts these "payoffs" for an ε "small enough".

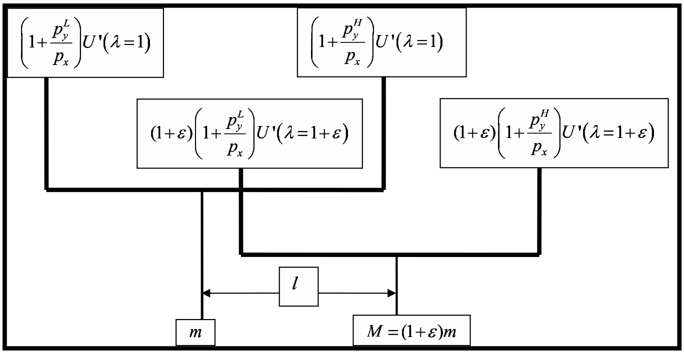

Figure 1. Contingent payoffs from an indexed bond

Notice, as arranged in the figure the payoffs are increasing from left to right. Since $U'(\lambda)$ is increasing in λ and $\varepsilon > 0$, $\left(1 + \frac{p_y^L}{p_x}\right) U'(1) < \left(1 + \frac{p_y^L}{p_x}\right) U'(1+\varepsilon)$ and $\left(1 + \frac{p_y^H}{p_x}\right) U'(1) < \left(1 + \frac{p_y^H}{p_x}\right) U'(1+\varepsilon)$. Since u

is continuous and $p_y^H > p_y^L$, for ε small enough $\left(1+\frac{p_y^L}{p_x}\right)U'(1+\varepsilon) < \left(1+\frac{p_y^H}{p_x}\right)U'(1)$. Now, to simplify matters dramatically, suppose $\nu^H = \nu^L = 0$. Hence, if the agent were to go long in the indexed bond the payoffs in events where the monetary shock is low and high are $\left(1+\frac{p_y^L}{p_x}\right)U'(1)$ and $\left(1+\frac{p_y^L}{p_x}\right)U'(1+\varepsilon)$, respectively. Similarly, if the agent were to go short in the indexed bond the payoffs in events where the monetary shock is low and high are $\left(1+\frac{p_y^H}{p_x}\right)U'(1)$ and $\left(1+\frac{p_y^H}{p_x}\right)U'(1+\varepsilon)$, respectively. Hence, the most (in terms of the relative price $\frac{q^i}{q^n}$) the agent would want to bid for an unit of the indexed bond is $\left(1+\frac{p_y^L}{p_x}\right)U'(1+\varepsilon)$. On the other hand the minimum the agent would ask for going short on an indexed bond is $\left(1+\frac{p_y^H}{p_x}\right)U'(1)$. Hence, there must be a non-degenerate interval of prices at which the agent strictly prefers to maintain a zero holding of the indexed bond. Notice, as depicted in the figure, the effect of increasing ε would be to increase the distance l, since both $\left(1+\frac{p_y^L}{p_x}\right)U'(1+\varepsilon)$ and $\left(1+\frac{p_y^H}{p_x}\right)U'(1+\varepsilon)$ will rise relative to the other payoffs. This implies that, for ε large enough, the portfolio inertia interval will collapse. Notice, it is not necessary that $\mathcal{A} = 1-\nu^H-\nu^L = 1$ for the portfolio inertia interval to emerge. A portfolio inertia interval will exist for $\mathcal{A} < 1$, as long $\mathcal{A}$ is high enough. Since the ranking of states (according to consumption) reverses when switching between a long position and a short position, the "relevant" probability also switches between evaluating a long and a short position. It were as if when evaluating a long position the priors are "concentrated" on states 1 and 2, while they are concentrated on 3 and 4 when evaluating a short position. It is this feature which causes a "kink" in the utility functional at the zero holding position and leads to the portfolio inertia interval.

Remark 1. An analogous argument, mutatis mutandis, shows that even for a lender (i.e., $S < 0$) one would obtain a portfolio inertia interval. While reconstructing the argument, essentially, the only adjustment one needs to make is to bear in mind is that for a lender $U'(\lambda)$ is decreasing in λ and hence the "ordering" of the states is reversed (compared to the ordering in the case of a saver).

Remark 2. One could also wonder whether there is a range of prices at which there is a zero holding of the nominal bond. Indeed, in the set up we have described so far, if we were to have, in addition, ambiguous beliefs about the inflation (i.e., $\mu(\{1,3\}) + \mu(\{2,4\}) < 1$), then for a high enough level of *this ambiguity, by an argument analogous to the one given above, there will be an interval of relative bond prices,* $\frac{q^i}{q^n}$*, at which the agent will only have a zero holding of the nominal bond. However, this is not very compelling as the result is not robust in an important way. It does not hold any more if the agent were to have some second period income, preset in nominal terms, that is not derived from bond holdings.*

Imagine for instance that the agent receives a state contingent income stream, $\{\overline{m}^s\}_{s=1}^4$*, preset in nominal terms. This could represent any previously contracted arrangement like, wage income, pension, social security benefits, etc., that are no more than partially indexed (i.e., they have a nominal component). In this "richer" model, the second period budget constraints are as follows:*

$$\begin{aligned}
x^1 &= \bar{x} + \frac{\left(p_x + p_y^H\right)}{q^i p_x} S + \left(1 - \left(p_x + p_y^H\right)\frac{q^n}{q^i}\right)\frac{b^n}{p_x} + \frac{\overline{m}^1}{p_x} \\
x^2 &= \bar{x} + \frac{\left(p_x + p_y^H\right)}{q^i p_x} S + \left(\frac{1}{\lambda} - \left(p_x + p_y^H\right)\frac{q^n}{q^i}\right)\frac{b^n}{p_x} + \frac{\overline{m}^2}{\lambda p_x} \\
x^3 &= \bar{x} + \frac{\left(p_x + p_y^L\right)}{q^i p_x} S + \left(1 - \left(p_x + p_y^L\right)\frac{q^n}{q^i}\right)\frac{b^n}{p_x} + \frac{\overline{m}^3}{p_x} \\
x^4 &= \bar{x} + \frac{\left(p_x + p_y^L\right)}{q^i p_x} S + \left(\frac{1}{\lambda} - \left(p_x + p_y^L\right)\frac{q^n}{q^i}\right)\frac{b^n}{p_x} + \frac{\overline{m}^4}{\lambda p_x}
\end{aligned}$$

Now, it follows immediately, except in the very specific case wherein $\left(\overline{m}^2, \overline{m}^4\right) = \lambda\left(\overline{m}^1, \overline{m}^3\right)$*, having a zero holding of nominal bonds does not allow the agent to be rid of the inflation risk. Indeed,* $b^n = 0$ *implies* $x^1 = x^2$ *and* $x^3 = x^4$ *if and only if* $\left(\overline{m}^2, \overline{m}^4\right) = \lambda\left(\overline{m}^1, \overline{m}^3\right)$*. This, of course, is the case only if* <u>all</u> *income is* <u>fully</u> *indexed or there is no preset nominal income* $(\overline{m}^s = 0, \forall s)$*. Hence, whenever,* $\left(\overline{m}^2, \overline{m}^4\right) \neq \lambda\left(\overline{m}^1, \overline{m}^3\right)$ *and* $b^n = 0$ *consumption is* <u>strictly</u> *ordered across states 1 and 2 and across states 3 and 4. (For instance, if* $\overline{m}^s = \overline{m} > 0$ $\forall s$*, one has* $x^1 > x^2$ *and* $x^3 > x^4$*.) This strict ordering is preserved in the* ϵ*-neighborhood* $b^n \in (-\epsilon, \epsilon)$*. Hence, there is no switch in the probability the DM "applies" when evaluating going short and going long on the nominal bond. Thus the usual expected utility logic applies and the derivative (of the utility functional) is continuous at the point of zero holding, implying that there is no non-degenerate interval of (bond) prices at which the agent holds zero nominal bond.*

Remark 3. We have assumed constant x *endowment. This is essentially for expositional ease and to make the model as simple as possible. If one were to introduce uncertain endowment of good* x*, a similar reasoning would hold: given a value of the endowment, there would still be two other (orthogonal) sources of uncertainty, namely, the price of good* x *and the value of the indexed bundle (the price of good* y*). While it is true that Dow and Werlang's result assumed that the agent's endowment was riskless, a crucial contribution of Epstein and Wang (1994) was to show that the result could be generalized to the case of a non-stochastic endowment so long as there were a part of the asset payoff that was orthogonal to the endowment. If one were to introduce* *say two values for the level of endowment (say, $\bar{x}, \underline{x}$), the actual number of states would be eight, and for sufficiently high ambiguity aversion, the agent would not be willing, for an interval of prices, to use the indexed bond to hedge the risk linked to his x-endowment.*

4 No-trade in indexed bonds: a general equilibrium framework

4.1 The institutional setup

The previous section shows that for both types of agents, those who save and those who borrow, there exists a range of relative bond prices, corresponding to each agent, at which the agent maintains a zero holding in indexed bonds. However, this does not immediately translate into a conclusion about conditions under which no-trade in indexed bonds is the *unique equilibrium* outcome. To be able to get to that conclusion several questions remain to be answered. What would ensure that the bid-ask price intervals of the various agents "overlap"? Why should equilibrium bond price fall within the zone of "overlap"? Further, since we know that for a portfolio inertia interval to emerge the goods prices have to vary across states in particular ways, a related question is are such state-contingent price variations consistent with competitive equilibrium in money, goods and bond markets? To deal with such issues we turn next to a two-period monetary general equilibrium model, general in the sense that *all* prices are obtained endogenously by (simultaneous) market clearing in bond, goods and money markets. Since the overall aim is to lay out the logic of no trade as transparently as possible, we have chosen the simplest model we could. For instance, given the crucial role of the movement of goods prices in obtaining no-trade, the relationship between such prices and the parameters of the model has been kept as tractable as possible. Arguably, the more realistic source of sectoral price movements are shifts in preferences and/or technological shocks. However, the analysis here is exposited within a framework of the simplest general equilibrium model known to economists, an exchange economy without any production, wherein relative-price movements are derived by perturbing endowments. The point, we emphasize, is transparency, not realism *per se*.

There are two groups of agents in the model. The first group (whose agents are indexed by $h = 1, \ldots, H$) are those who trade on financial markets, while the second group (whose agents are indexed by $k = 1, \ldots, K$) has no access to any financial markets and therefore all the agents in this group consume all the revenue from their endowment spot by spot. There are three goods in this economy, x, y, and z. Agents h consume only goods x and z while agents k consume only goods y and z. We also assume that agents h have real endowments only in goods x and z, while agents k have real endowments only in goods y and z. In addition, agents h may have nominal endowments.

Nominal endowments are any precontracted transfers, positive or negative, between agents that are (at least partly) set in nominal terms. Examples include, wages and salaries, house rents or even something like alimony or child support payments. (In the U.S. alimony and child support payments are almost never indexed, see (Shiller, 1997).)

To see the rationale of "type-casting" agents as above recall, from what was noted in the introductory section, we want to ensure in the model that with respect to *any* two agents wishing to trade in indexed bonds, it is true that the indexation bundle contains at least one good which is not consumed by either of the agents. This condition, of course, would not be satisfied if an h-type agent were to trade bonds with a k-type agent. We have each type of agent consuming two goods rather than one, unlike in the model in the previous section, so that there may be market exchange among agents, thereby obtaining well-defined prices at equilibrium (reflecting the common utility gradients). Informally put, the focus of the "show" will be the intertemporal exchange between the h-type agents, with the role of k-type agents being essentially that of a necessary "prop", enabling the determination of the relative price of the good not figuring in the consumption baskets of agents trading bonds.

There are two periods in the model; uncertainty essentially comes into play in the final period. The endowment of h-type agents is uncontingent, given by $((\bar{x}_h^0, \bar{z}_h^0), (\bar{x}_h, \bar{z}_h, \overline{m}_h))$, where $(\bar{x}_h^0, \bar{z}_h^0)$ is the endowment in the initial period, Period 0, and $(\bar{x}_h, \bar{z}_h, \overline{m}_h)$ is the endowment in the final period, Period 1. $\overline{m}_h$ denotes the nominal endowment, so that $\overline{m}_h \lesseqqgtr 0$ and since transfers should balance across households, we have

$$\sum_{h=1}^{H} \overline{m}_h = 0.$$

Note though, the endowments vary across households; this heterogeneity is the reason why h-type agents trade intertemporal transfers. The endowment of k-type agents are given by $(\bar{y}_k^0, \bar{z}_k^0)$ in the initial period[8]. Their final period endowment in good z is uncontingent and equal to $\bar{z}_k$. We assume, though, their endowments in good y *is* contingent: in state t, $\bar{y}_k^t$, are such that $\sum_{k=1}^{K} \bar{y}_k^t = y^L$ for, say, $t = 1, \ldots, \tau$ and $\sum_{k=1}^{K} \bar{y}_k^t = y^H$ for $t = \tau + 1, \ldots, T$. Thus, in terms of total endowments, there are two "aggregate" states: one where the total endowment of good y is low (y^L) and another, where the total endowment of y is high (y^H). As will be seen, it is this variation in aggregate endowment which completely determines the variation in the relative price of y.

[8]Note, h-type agents do not have nominal endowments in the initial period. k-type agents do not have nominal endowments at all. This is just to save on notation; introducing such endowments would not make the slightest difference to any of our results.

There is also (outside) money in the model, whose supply in the Period 0 is fixed at M^0 but may take on two values in the Period 1, m or M, where $M \equiv \lambda m$, $\lambda > 1$. The role of money is simply to facilitate exchange. Hence, at each spot, we assume the standard fiction that agents sell to a central authority *all* their endowments against currency issued by the central authority and then buy back from that authority the goods they want to consume (see (Magill and Quinzii, 1992)). The money obtained from the central authority by agent h (respectively, k) from the sale of endowments in state s is denoted m_h^s (respectively, m_k^s).

Uncertainty in the model is exhaustively represented by the state space

$$\mathcal{S} \equiv \{0\} \cup \{\{1, ..., T\} \times \{m, M\}\},$$

where, $\{0\}$ refers to the only state of the world in Period 0, $\{1, ..., T\}$ indexes contingencies in Period 1 obtaining due to variation in real endowments of agents, $\{m, M\}$ indexes the variation in money supply. Let $s \in \mathcal{S}$ be an index for states, $s = 0, 1, \ldots, S$. We denote the (absolute) prices of goods x, y, and z as p_x^s, p_y^s, and p_z^s, respectively, in state s.

There are two financial assets in the model, traded in Period 0. The first is a nominal bond, b^n, that pays off one unit of money in all states and with its price denoted q^n. The second is an indexed bond, b^i, that pays off a bundle of goods at each state in Period 1. We take this bundle to be state-independent and comprising of a unit of each good traded in the economy. Hence, the monetary return to holding a unit of this indexed bond is $p_x^s + p_y^s + p_z^s$ in state $s = 1, \ldots, S$. Its price is denoted q^i.

For the moment, denote an agent h's preferences by a functional $V_h\left((x_h^0, z_h^0), \ldots, (x_h^S, z_h^S)\right)$, on which we'll impose assumptions detailed later on. His maximization problem is hence:

$$\begin{array}{l} Max_{x_h, z_h, b_h^i, b_h^n} V_h\left((x_h^0, z_h^0), \ldots, (x_h^S, z_h^S)\right) \\ s.t. \left\{ \begin{array}{rl} p_x^0 \bar{x}_h^0 + p_z^0 \bar{z}_h^0 &= m_h^0 \\ p_x^0 x_h^0 + p_z^0 z_h^0 &= m_h^0 - q^i b_h^i - q^n b_h^n \\ p_x^s \bar{x}_h^s + p_z^s \bar{z}_h + \bar{m}_h &= m_h^s \\ p_x^s x_h^s + p_z^s z_h^s &= m_h^s + b_h^n + (p_x^s + p_y^s + p_z^s) b_h^i \ , \ s = 1, \ldots, S \end{array} \right. \end{array}$$

Agents k, who have no access to financial markets, have to solve $S + 1$ separate maximization programs. We assume that their preferences at each spot take the simple form of a Cobb-Douglas function: $(y_k^s)^\alpha (z_k^s)^{1-\alpha}$ for $\alpha \in (0, 1)$. Hence, their maximization problem for $s \in \mathcal{S}$, is:

$$\begin{array}{ll} Max_{y_k^s, z_k^s} & (y_k^s)^\alpha (z_k^s)^{1-\alpha} \\ s.t. & \left\{ \begin{array}{l} p_y^s \bar{y}_k^s + p_z^s \bar{z}_k = m_k^s \\ p_y^s y_k^s + p_z^s z_k^s = m_k^s \end{array} \right. \end{array}$$

An equilibrium of this model is therefore an allocation $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{m}, \mathbf{b}^i, \mathbf{b}^n)$ and prices $(p_x, p_y, p_z, q^i, q^n)$ such that, given these prices agents solve their maximization problems and markets clear.

Observe that money as we introduced it is simply a veil and we can rewrite the budget constraints as follows, for agent h:

$$\begin{cases} p_x^0 x_h^0 + p_z^0 z_h^0 = p_x^0 \bar{x}_h^0 + p_z^0 \bar{z}_h^0 - q^i b_h^i - q^n b_h^n \\ p_x^s x_h^s + p_z^s z_h^s = \overline{m}_h + p_x^s \bar{x}_h^s + p_z^s \bar{z}_h + b_h^n + (p_x^s + p_y^s + p_z^s) b_h^i \quad s = 1, \ldots, S \end{cases}$$

and, for agent k in state s $(s = 0, 1, \ldots, S)$:

$$p_y^s y_k^s + p_z^s z_k^s = p_y^s \bar{y}_k^s + p_z^s \bar{z}_k$$

One can also use the particular structure of the model to simplify the market clearing condition for good z. Indeed, adding the budget constraints in state s of agents h, one gets, at equilibrium, $\sum_{h=1}^{H} z_h^s = \sum_{h=1}^{H} \bar{z}_h$ (under the assumption that $p_x^s > 0$, which is met since preferences are assumed strictly monotonic). Similarly, for agents k, one obtains, from adding their budget constraints in state s and using equilibrium condition on the market for good y (plus the fact that, at an equilibrium, $p_z^s > 0$), that $\sum_{k=1}^{K} z_k^s = \sum_{k=1}^{K} \bar{z}_k$. Thus, the market clearing conditions on the market for good z can be split in two equalities as follows:

$$\sum_{h=1}^{H} z_h^s = \sum_{h=1}^{H} \bar{z}_h \text{ and } \sum_{k=1}^{K} z_k^s = \sum_{k=1}^{K} \bar{z}_k \quad s = 0, \ldots, S$$

Hence, the market for good z can be "divided in two", agents h exchanging among themselves, and similarly for agents k. The intuition for this is fairly obvious once one lifts the "veil of money" and considers the nature of "real" exchange in the model. The point is, given that the two types of agents share only one good between their respective consumption baskets, there cannot be any "real" exchange between these groups on spot markets.

Finally, notice that the market clearing condition on the money market can be written as

$$p_x^s \sum_{h=1}^{H} \bar{x}_h + p_y^s \sum_{k=1}^{K} \bar{y}_k^s + p_z^s \left(\sum_{h=1}^{H} \bar{z}_h + \sum_{k=1}^{K} \bar{z}_k\right) = M^s \quad s = 0, \ldots, S$$

while the market clearing condition on the bond markets are $\sum_{h=1}^{H} b_h^i = \sum_{h=1}^{H} b_h^n = 0$.

4.2 Equilibrium prices in goods markets

We can further reduce the model by noticing that only aggregate states "matter". Indeed, note that there are two sources of (aggregate) uncertainty in this model: one is linked to the money supply, the second stems from the randomness in the (aggregate) endowment in good y of agents k. As we will be only interested in the equilibrium allocation of the h agents (and in particular whether they hold indexed bonds or not), the only way this last source of uncertainty is relevant to h agents is through the effect it has on prices. Now, observe that we can solve for the equilibrium relative price of y with respect to z, spot by spot. Indeed, agents k demand functions are easily computed and are equal to:

$$y_k^s(p_y^s, p_z^s) = \alpha \frac{p_y^s \bar{y}_k^s + p_z^s \bar{z}_k}{p_y^s} \text{ and } z_k^s(p_y^s, p_z^s) = (1-\alpha) \frac{p_y^s \bar{y}_k^s + p_z^s \bar{z}_k}{p_z^s}$$

Hence, at equilibrium,

$$\frac{p_y^s}{p_z^s} = \frac{\alpha}{1-\alpha} \frac{\sum_{k=1}^K \bar{z}_k}{\sum_{k=1}^K \bar{y}_k^s}$$

and therefore, the ratio of the prices p_y^s and p_z^s depends only on the *aggregate* (among k agents) endowments of good y and z, and thus, can take on only two values, whether aggregate endowment in y is high (y^H) or low (y^L). Note that the price levels do depend on the money supply. To sum up, we need, for our purposes, concentrate only on four states $\omega \in \Omega \equiv \{1, 2, 3, 4\}$ defined as follows[9] :

$\omega = 1$: low money supply (m) & low aggregate y-endowment (y^L)
$\omega = 2$: high money supply (M) & low aggregate y-endowment (y^L)
$\omega = 3$: low money supply (m) & high aggregate y-endowment (y^H)
$\omega = 4$: high money supply (M) & high aggregate y-endowment (y^H)

We now describe agent's h preferences, following partly a specification due to (Magill and Quinzii, 1997). At state 0, h-type agents' utility function is written as $u(x_h^0, z_h^0)$, where $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, concave, differentiable and homogeneous of degree 1. In state $\omega = 1, 2, 3, 4$, the spot utility function of agent h is given by $f_h(u(x_h^\omega, z_h^\omega))$, where $f_h : \mathbb{R} \to \mathbb{R}$ is strictly increasing, strictly concave and differentiable, and u is as defined above. The linear homogeneity of u will facilitate the tractability of the equilibrium (contingent) price function (essentially, the assumption ensures that goods prices, in each spot market, are independent of the distribution of wealth among agents) while the concavity of f_h is a simple way of endowing agents with risk aversion as well as a desire to smooth consumption across periods.

Type-h agents are endowed with (common) beliefs about the money supply process and the process generating the (aggregate) y-endowments. The capacity $\mu = (\mu^m, \mu^M)$ denotes their (marginal) belief about the money supply: μ^m is the (possibly non-additive) probability the money supply is m and μ^M is the probability that it is M; $\mu^m + \mu^M \leq 1$. Their (marginal) beliefs on the process generating the y-endowments (a good they do not consume and are not endowed with) are represented[10] by $\nu = (\nu^L, \nu^H)$, with $\nu^L + \nu^H \leq 1$.

[9]We denote these aggregate states by ω to distinguish them from the underlying states s which encompass some heterogeneity among k-type endowments. We'll continue to use the notation $\omega = 0$ to denote the first period.

[10]Note that, ν^H refers now to the state with high y-endowment, and therefore low p_y, while it referred to the high p_y in the previous section.

The overall beliefs on Ω is given by the independent product $\mu \otimes \nu$. Preferences of agent h are thus represented by the functional V_h, where,

$$V_h\left((x_h^0, z_h^0), \ldots, (x_h^4, z_h^4)\right) \equiv u(x_h^0, z_h^0) + \mathbb{CE}_{\mu\otimes\nu} f_h\left(u(x_h^\omega, z_h^\omega)\right).$$

Under our assumptions (essentially, the linear homogeneity of u) at an equilibrium, the maximization problem of each h agent can be decomposed to separate the financial decision about the allocation of resources across states from the decision about the consumption mix at each state. Turning first to the latter, given a stream $(R_h^0, R_h^1, \ldots, R_h^4)$ of income with $R_h^0 = p_x^0 \bar{x}_h^0 + p_z^0 \bar{z}_h^0 - q^i b_h^i - q^n b_h^n$ and $R_h^\omega = p_x^\omega \bar{x}_h^\omega + p_z^\omega \bar{z}_h + (p_x^\omega + p_y^\omega + p_z^\omega) b_h^i + b_h^n + \bar{m}_h$, $\omega = 1, \ldots, 4$, the agent solves the problem:

$$\begin{array}{cl} Max_{x_h^\omega, z_h^\omega} & u(x_h^\omega, z_h^\omega) \\ \text{s.t.} & p_x^\omega x_h^\omega + p_z^\omega z_h^\omega = R_h^\omega \end{array}$$

At the optimal choice, one gets for all $\omega = 0, 1, ..., 4$:

$$\frac{\nabla_1 u(x_h^\omega, z_h^\omega)}{\nabla_2 u(x_h^\omega, z_h^\omega)} = \frac{p_x^\omega}{p_z^\omega}$$

where $\nabla_i u(x_h^\omega, z_h^\omega)$ is the derivative of u with respect to its i^{th} component. By homogeneity of degree one, the gradients are collinear among agents only if their consumption vectors are collinear as well. Recall, at an equilibrium agents h only trade among themselves and do not trade with the k agents. Hence, each agent h's consumption in state ω is a fraction α_h^ω of total endowment of h-agents with

$$\begin{aligned} \alpha_h^0 &= \frac{p_x^0 \bar{x}_h + p_z^0 \bar{z}_h - q^i b_h^i - q^n b_h^n}{p_x^0 \sum_{h=1}^H \bar{x}_h^0 + p_z^0 \sum_{h=1}^H \bar{z}_h^0} \\ \alpha_h^\omega &= \frac{p_x^\omega \bar{x}_h + p_z^\omega \bar{z}_h + (p_x^\omega + p_y^\omega + p_z^\omega) b_h^i + b_h^n + \bar{m}_h}{p_x^\omega \sum_{h=1}^H \bar{x}_h + p_z^\omega \sum_{h=1}^H \bar{z}_h}, \quad \omega = 1, ..., 4 \end{aligned}$$

Hence, at an equilibrium,

$$(x_h^\omega, z_h^\omega) = \alpha_h^\omega \left(\sum_{h=1}^H \bar{x}_h, \sum_{h=1}^H \bar{z}_h \right)$$

Therefore, agent h's utility, at an equilibrium, can be rewritten in state $\omega = 1, ..., 4$, $u\left(x_h^\omega, z_h^\omega\right) = \alpha_h^\omega u$, where u is simply the utility at the endowment point, i.e., $u \equiv u\left(\sum_{h=1}^H \bar{x}_h, \sum_{h=1}^H \bar{z}_h\right)$. Observe that the same holds in the first period, i.e., $u\left(x_h^0, z_h^0\right) = \alpha_h^0 u^0$, with $u^0 \equiv u\left(\sum_{h=1}^H \bar{x}_h^0, \sum_{h=1}^H \bar{z}_h^0\right)$.

Finally, since relative prices of good x and z in state ω are equal to the gradient of an agent h's utility function, it is easy to see that

$$\frac{p_x^1}{p_z^1} = \frac{p_x^2}{p_z^2} = \frac{p_x^3}{p_z^3} = \frac{p_x^4}{p_z^4} \equiv \zeta$$

given that endowments of goods x and z are constant across states. In fact, it turns out that, the absolute price of goods x and z do not depend on the amount of good y available in the economy. In other words, and since there is no uncertainty on the total endowments of goods x and z, their price depends only on the money supply. This is the content of Proposition 1, below, which is proved in the appendix. A direct corollary is that the price of y, conditional on the monetary state, is completely determined by the aggregate endowment in y. Proposition 2 (proved in the appendix) which essentially shows that monetary equilibrium requires the price vector in state 2 (respectively, 4) is simply a λ-multiple of prices in state 1 (respectively, 3), completes the required characterization of equilibrium prices.

Proposition 1. *At an equilibrium,* $p_x^1 = p_x^3$, $p_x^2 = p_x^4$, $p_z^1 = p_z^3$, *and* $p_z^2 = p_z^4$.

From this proposition, it is easy to show that, at an equilibrium, $\frac{p_y^1}{p_y^3} = \frac{y^H}{y^L} = \frac{p_y^2}{p_y^4}$ (this follows from the fact that $\frac{p_y^1}{p_z^1} = \frac{p_y^3}{p_z^3}\frac{y^H}{y^L}$ and $p_z^1 = p_z^3$).

Proposition 2. *At an equilibrium,* $(p_x^1, p_y^1, p_z^1) = \frac{1}{\lambda}(p_x^2, p_y^2, p_z^2)$ *and* $(p_x^3, p_y^3, p_z^3) = \frac{1}{\lambda}(p_x^4, p_y^4, p_z^4)$.

The following table, then, summarizes the equilibrium prices, and the corresponding return from an unit of an indexed bond at each state ω, $\omega \in \Omega$. The reader will recall the table is identical (but for price of the good z) to the one presented in Section 3.

State ω	Prices	Return from an indexed bond
1	(p_x, p_y^H, p_z)	$p_x + p_y^H + p_z$
2	$(\lambda p_x, \lambda p_y^H, \lambda p_z)$	$\lambda \times (p_x + p_y^H + p_z)$
3	(p_x, p_y^L, p_z)	$p_x + p_y^L + p_z$
4	$(\lambda p_x, \lambda p_y^L, \lambda p_z)$	$\lambda \times (p_x + p_y^L + p_z)$

4.3 The nature of equilibrium in bond markets

We now turn to the intertemporal maximization problem of agent h and derive the principal formal conclusions of our analysis. The first result, which is in the nature of a benchmark, shows that, for generic endowments, if beliefs are not ambiguous, there will always be trade in indexed bonds. The second and main result shows that, at equilibrium, if ambiguity of belief about the $y-$prices ($\mathcal{A}(\nu) \equiv 1 - \nu^L - \nu^H$) is large enough and inflation risk (λ) is not too high, the indexed bond is not traded and only the nominal bond is traded. As we show, this result holds irrespective of the degree of ambiguity about the money supply. In what follows, we first explain an intuition of the equilibrium reasoning underlying the results and then state the theorems, with the formal proofs appearing in the appendix.

We begin by considering the nature of the equilibrium in the indexed bond market at two values of λ, $\lambda = 1$ and $\lambda = 1 + \varepsilon$, where ε is a positive number arbitrarily close to 0. Consider, first, the case wherein $\lambda = 1$. Without any inflation risk at all, clearly, all borrowing and lending will be done through nominal bonds, at equilibrium. Take two h-type agents, h' and h who save and borrow, respectively, in the initial period. Their utility in the final period, with slight abuse of the notation, may be written as $f_{h'}\left(u(\bar{x}_{h'}, \bar{z}_{h'}; \frac{S_{h'}}{\lambda q^n})\right)$ and $f_h\left(u(\bar{x}_h, \bar{z}_h; \frac{S_h}{\lambda q^n})\right)$, where $S_{h'} > 0$ and $S_h < 0$ denote the amount saved and the amount borrowed by h' and h respectively. Define the "marginal utilities", $U'_h(\lambda) \equiv f'_h\left(u(\bar{x}_{h'}, \bar{z}_{h'}; \frac{S_h}{\lambda q^n})\right)$ and $U'_{h'}(\lambda) \equiv f'_{h'}\left(u(\bar{x}_{h'}, \bar{z}_{h'}; \frac{S_{h'}}{\lambda q^n})\right)$. Notice, $U'_{h'}(\lambda) \uparrow$ in λ while $U'_h(\lambda) \downarrow$ in λ, since $S_{h'} > 0$ and $S_h < 0$; intuitively, inflation affects the welfare of savers and borrowers differently. Furthermore, it must be necessarily true at an equilibrium that $U'_h(\lambda = 1) = U'_{h'}(\lambda = 1)$. This is so since if there is no inflation risk, h-agents are effectively trading in a complete market when trading only in nominal bond, and thus the equilibrium is Pareto optimal.

First suppose, ceteris paribus, there were no ambiguity, i.e., $1 - \nu^L - \nu^H = 0$ and $1 - \mu^m - \mu^M = 0$, so that the DM's behavior were that of an SEU agent. At $\lambda = 1$, the "utility return" from an (infinitesimal) unit of an indexed bond at equilibrium must be $U'_h(\lambda = 1) \times \mathbb{E}\left(p_x + p_y^\omega\right) = U'_{h'}(\lambda = 1) \times \mathbb{E}\left(p_x + p_y^\omega\right) \equiv q^i(\lambda = 1; h, h')$. Putting it differently,
$q^i(\lambda = 1; h, h')$ is the price at which the agents (h and h') are indifferent between not trading and trading an infinitesimal amount of indexed bonds. Similarly, for an arbitrary λ, define $q^i(\lambda; h)$ (respectively, $q^i(\lambda; h')$) as the minimum (maximum) price h (h') is willing to accept (pay) to trade in the indexed bond. Next, consider a perturbation of λ to $\lambda = 1 + \varepsilon$. Recalling the effect of a change in λ on $U'_h(\lambda)$ and $U'_{h'}(\lambda)$, it is straightforward to see that

$$\begin{aligned} q^i(\lambda = 1 + \varepsilon; h') &\equiv U'_{h'}(\lambda = 1 + \varepsilon) \times \mathbb{E}\left(p_x + p_y^\omega\right) \\ &> q^i(\lambda = 1; h, h') \\ &> U'_h(\lambda = 1) \times \mathbb{E}\left(p_x + p_y^\omega\right) \geq q^i(\lambda = 1 + \varepsilon; h). \end{aligned}$$

Intuitively, since the saver is affected adversely by inflation, relative to the debtor, the indexed bond is more valuable to the saver in the presence of inflation, and also, more valuable than it is to the debtor. Hence, inevitably, with inflation risk creeping up there will be gains from trading in indexed bonds and indexed bonds will be traded at equilibrium under SEU. This is depicted in Figure 2, below, in the left-hand-side panel and formally stated in our first theorem[11].

[11]Both theorems refer to properties that hold "generically". The term is applied in a way that is now standard in economic theory. Notice, endowments are points in

Theorem 1. *Suppose, $\mu^m + \mu^M = 1$ and $\nu^L + \nu^H = 1$. Then, for generic first period aggregate endowments, there is trade in the indexed bond whenever $\lambda \neq 1$.*

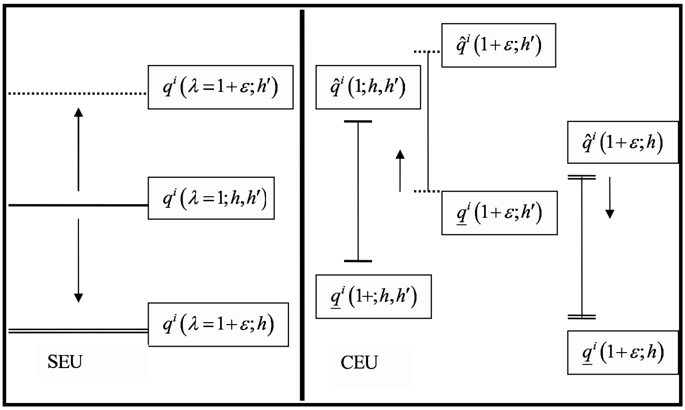

Figure 2. Equilibrium in the indexed bond market

Next suppose, agents have CEU preferences and beliefs about the y-prices are ambiguous, i.e., $1 - \nu^L - \nu^H > 0$. As before, we first consider the equilibrium at $\lambda = 1$. As would be evident from our discussion in Section 3, there would exist a portfolio inertia interval: there will be a bid price corresponding to (perceived) marginal gain from moving (infinitesimally) into a indexed bond, and an ask price, that is strictly lower, corresponding to the (perceived) marginal gain from going short on the indexed bond. The bid-ask interval will correspond to an interval of expected marginal utilities, where the lower end of the interval is evaluated by applying the probability measure that minimizes the expectation for an agent going long and the upper end is evaluated by applying the probability measure that minimizes the expectation for an agent going short:

Next, consider the equilibrium given the perturbation $\lambda = 1 + \varepsilon$. Noting again the effect of the perturbation on $U'_h(\lambda)$ and $U'_{h'}(\lambda)$, it is straightforward to see that for h', the saver, the *entire* interval moves *up*, whereas for h, the debtor, the *entire* interval moves *down.* The extent of movement is greater, the greater increase in λ. Hence, for $\lambda > 1$, but small enough, the intervals overlap and the bid price of the saver remains (strictly) lower than the ask price required by the lender:

$\mathbb{R}^6$ and the μ-beliefs are simply points on the 2-dimensional simplex. We say that a property is satisfied for generic endowments (respectively, μ's) if, for every endowment (respectively, μ) vector there is an open dense neighborhood of endowment (respectively, μ) vectors that generate economies that satisfy this property. Thus, if a property is satisfied for all generic endowments (respectively, μ's), small perturbations of the endowment (respectively, μ) in any economy can generate a new economy that satisfies this property robustly, even if the original economy does not.

$$\underline{q}^i (1+\varepsilon; h') \equiv U'_{h'} (1+\varepsilon) \times \underline{\mathbb{E}} \left(p_x + p_y^{\omega}\right) < U'_h (1+\varepsilon) \times \bar{\mathbb{E}} \left(p_x + p_y^{\omega}\right) \\ \equiv \hat{q}^i (1+\varepsilon; h)$$

This is represented in Figure 2 in the right-hand side panel. It is also evident in the figure that if the increase in λ were large enough then the intervals move apart enough not to overlap. Hence, we have, for λ small enough, there is no trade in indexed bonds at equilibrium. This is formally stated in the theorem below.

Finally, recalling Remark 2, if there is at least one agent with a positive nominal endowment (and hence, at least one other agent with a negative nominal endowment) it follows that there is at least a pair of agents who do not optimally choose a zero-holding of the nominal bond over a non-degenerate interval of relative bond prices. Hence, for these agents the situation is no different from the SEU case described earlier (and depicted on the left-hand-side panel of Figure 2). "Typically" such agents would want to trade in nominal bonds. The discussion is summarized by the two parts of Theorem 2, below.

Theorem 2. *Suppose, $\mu^m + \mu^M \leq 1$ and $\nu^L + \nu^H < 1$. Then, there exists a bound δ, $\delta > 1$, such that, if $\lambda < \delta$, there exists γ, $0 < \gamma < 1$, such that if $\mathcal{A}(\nu) > \gamma$, then at an equilibrium,*

(a) the indexed bond is not traded, i.e., $b_h^i = 0$ for all h,
(b) for generic μ-beliefs, there is trade in the nominal bond as long as there exists an h such that $\bar{m}_h > 0$.

Remark 4. Notice, both parts of the theorem hold regardless of the level of ambiguity w.r.t. the μ-beliefs. Indeed, the formal proof is a lot shorter (and simpler) if one were to assume that μ-beliefs were unambiguous. We, however, do not impose this restriction in the analysis so that one may obtain a more informed idea of the nature of robustness of the result. Of course, nominal endowments would no longer play a role in the argument if μ-beliefs were assumed to be unambiguous.

Remark 5. Parameter λ refers to inflationary risk: recall that it is equal to the ratio M/m, that is, the relative inflation in the "high money supply" state with respect to the "low money supply state". Since money is not a store of value in this model, the only risk related to second period money supply is indeed that of differential inflation in the various states of the world, which is precisely what is picked up by λ. Thus, the theorem states that if this risk is sufficiently small then, ambiguity about relative price movement prevents trade in indexed assets.

Remark 6. The proof of the theorem, in the Appendix, provides an explicit characterization of the bound δ in terms of the fundamentals of the model. As would have been intuitively evident from the discussion preceding the theorem, the degree to which inflation risk would be "tolerated" before indexed bonds *are traded depends (positively) on the "variability" of the relative prices. For instance, the* δ *is equal to* $\frac{2+\sum_{h=1}^{H}\bar{x}_h/y^L}{2+\sum_{h=1}^{H}\bar{x}_h/y^H}$ *when all agents (h and k) have Cobb-Douglas* $(.5,.5)$ *as their respective* $u(.,.)$ *functions.*

Remark 7. The logic underlying the result in Theorem 2(b) is actually instructive, indirectly, as to why ambiguity about the price movements of goods not *in the(h-agents') consumption basket was the crucial factor in obtaining no-trade in indexed bonds. Putting it differently, if we allowed the absolute prices of say,* x*, to vary in response to supply shocks and assumed agents had ambiguous beliefs about such price movements, that would not obtain the no-trade in indexed bonds (without ambiguity about price of* y*). The reasoning here is analogous to the one showing that the presence of nominal endowments precludes no-trade in nominal bonds. Since* x *is present in the endowment and/or affects utility directly (of* h*-agents), maintaining a zero position on the indexed bond would not get rid of the risk due to the variability of the price of* x*. Consequently, the ordering of states would not switch at the zero holding position. This is why we need the "prop" agents in the model: they are the ones who are the source of volatility of the price of good* y*, which is the crucial factor underlying the result. If we dropped these prop agents we would be taking away the* y*-good and therefore, the risk in an indexed asset orthogonal to the asset traders' endowment and consumption. If the asset did not contain this idiosyncratic risk, there would be trade in the two bonds, much like in a SEU economy (for further clarification see the discussion related to Figure 1 on page 887 and Example 2 in Mukerji-Tallon (2001)).*

Remark 8. In the real world inflation and relative price movements are correlated. Recall, though, the model assumes that the processes generating the money supply and the real shocks (to the aggregate endowment of y*) are believed to be independent. However, what is really crucial is that the process generating* p_y *must have at least one component that is believed to be orthogonal to the money supply: i.e., there has to be at least two states of the world across which the price of* y *changes even though the money supply stays constant. In other words, for the reasoning to hold, prices cannot be perfectly correlated with the money supply. Similarly, the logic behind our result does not require that* p_y *and* p_x *(or* p_z*) have to be independent, but that they are not perfectly correlated.*

5 Concluding discussion

The discussion in this section is in two parts. In the first part we relate the results obtained in this paper to those in the relevant (theoretical) literature. The second part considers to what extent the analysis here may be thought to explain the empirics of trade in indexed debt any more than the explanations already advanced. Implications of the results in terms of policy required to encourage trade in indexed bonds is also discussed.

5.1 Related (theoretical) literature

We begin by linking the result here to the findings in the literature applying ambiguity aversion to financial markets. Then we discuss the theoretical literature in the standard Savage paradigm that seeks to explain observed facts relating to trade in indexed debt.

As has already been noted, (Dow and Werlang, 1992) showed that a zero position may be held on a price interval if the agent's endowments were riskless. Obviously, an economy where all agents' endowments were unvarying across *all* states the question of asset trading and risk sharing is an uninteresting question. (Epstein and Wang, 1994) significantly generalized the (Dow and Werlang, 1992) result to find that price intervals supporting the zero position occurred (in equilibrium) if there were *some* states across which asset payoffs differ while endowments remain identical; in other words, asset payoffs have component of idiosyncratic risk. However, the focus of (Epstein and Wang, 1994) was the issue of asset pricing. In their model endowments are Pareto optimal, and consequently, the issue of whether ambiguity aversion cause assets not to be traded is not examined. (Mukerji and Tallon, 2001), building on the results in the two papers cited, finds conditions for an economy wherein the agents' price intervals overlap in such a manner such that *every* equilibrium of the economy involves no trade in an asset, and more importantly, conditions under which ambiguity aversion *demonstrably* "worsens" risk sharing and incompleteness of markets. One of the conditions, the presence of idiosyncratic risk, identified in (Mukerji and Tallon, 2001), is essentially the same as in the result of (Epstein and Wang, 1994) explained above. As has been suggested, it is possible to see that, for h-type agents, payoffs of indexed bonds contain an element of idiosyncratic risk derived from the risk inherent in the relative price of y. This is, essentially, how the finding in this paper links up with the results in the papers cited above.

The paper closest to ours, within the Savage paradigm, which seeks to explain the lack of indexed debt is (Magill and Quinzii, 1997). That paper compares the welfare improvements obtained from introducing within an incomplete markets setting, *in turn*, a nominal bond and an indexed bond. The welfare improvements derive from, essentially, the increase in the span of available assets (or, in other words, the "lessening" of incompleteness) that comes about due to the introduction of each type of bond. The more relevant result is that the welfare gain from introducing the indexed bond may be less (respectively, more) than that from introducing a nominal bond if the inflation risk was "small" (respectively, large) compared to the relative price risk. In contrast to the analysis in this paper, (Magill and Quinzii, 1997) does not actually obtain a equilibrium with no-trade in indexed bonds; indeed, as we confirm in Theorem 1, Savage rational agents will necessarily trade in indexed bonds as long as there is some inflation. Also, (Magill and Quinzii, 1997) do not allow *both* indexed and nominal bonds to be available for trade simultaneously; one or the other is available.

Another paper, within the Savage paradigm, which studies a similar question, but in a rather different framework is (Freeman and Tabellini, 1998). The framework in that paper is an overlapping generations model. The paper finds that nominal contracts are optimal in the presence of price level shocks if all the contracting parties have the same degree of constant relative risk aversion. With respect to relative price shocks, the optimality of nominal contracts follows from the fact that, in their model, agents cannot insure each other against this risk as they all are ex ante identical with respect to such shocks.

The present paper also complements the finding in Mukerji and Tallon (2004) which shows that ambiguity aversion (with CEU preferences) may help to explain why we see so little wage indexation. Among other things, the framework in that paper is one of bilateral contracting, not a general equilibrium market environment like it is here. Hence, the result there does not follow from the result here even. However, as we understand it, the same intuition explains both results, a point that is significant in so far as it shows that the intuition is robust across seemingly different trading environments. Finally, the paper adds to the growing literature on the economics applications of the idea of ambiguity aversion (see Mukerji and Tallon (2003b) for a survey).

5.2 Explaining the empirics of trade in indexed debt

Recall the intuition underlying the main result. Taking a long or a short position on the indexed bond implies betting on or against the (ambiguous) event wherein the (relative) price of good y will be high. To decide whether to bet on or against a particular event one has to reach a fine judgement about the relative likelihood of the event compared to its complement. Hence, the attraction of the zero holding position to the ambiguity averse agent. Moving from the zero position, in either direction, requires a compensating "ambiguity premium". Hence, the portfolio inertia intervals for the indexed bond. At low levels of inflation the bid-ask intervals of the borrower and the saver overlap and agents only trade in the nominal bond. As inflation rises, the saver is affected adversely while the borrower is made better off. As a consequence, the most the saver is willing to pay for the indexed bond goes up and the minimum the borrower would ask decreases. Hence if inflation were high enough, agents do trade indexed bonds. We also argued that, so long as agents held (non-zero) nominal endowments, this reasoning does not apply quite symmetrically to trade in nominal bonds. For instance, with a positive holding of the nominal bond, one would be betting on the event that (average) price level will be low. But, if one had, say, positive nominal endowments, moving to a zero holding would still imply one is betting in favor of the event that the price level will be low. Indeed, this would continue to be true even if one were to move marginally into a negative holding of the nominal bond. Hence, the portfolio inertia interval no longer occurs at the zero holding and trade occurs. Of course, by the same token, if some *endowments* were indexed, the argument for no trade in indexed bonds will be affected similarly.

Thus, according to the theory presented in this paper, it is the comparative lack of information about relative, as opposed to average, price movements, the comparative preponderance of nominal, as opposed to indexed, endowments that explains why trade in indexed bonds is observed only in exceptional circumstances but trade in nominal bonds is so widespread. While these are testable hypotheses that could provide the basis for a specific empirical investigation, that is a matter for future research. However, a lot that we do know about trade in indexed bonds is broadly consistent with the theory. Arguably, the theory is very consistent with the fact that typically indexed bonds are traded almost exclusively under extreme inflationary circumstances. Also, while trade in indexed bonds is negligible in most non-inflationary economies, it is more than negligible (though still quite small) in the few such economies where, in addition, there are some instances of indexed endowments, statutory wage indexation, as is the case, for example, in the U.K. and in Israel (statutory wage indexation in a limited number of sectors of the economy). Indeed, the reasoning predicts that one would observe a kind of hysteresis in the market for indexed bonds. In economies with inflationary past, where indexed bonds were traded when high inflation reigned, indexed bonds would continue to be traded even after inflation has been brought down to moderate levels because of the presence of indexed bonds as endowments. Perhaps, this explains the continued trade in indexed financial instruments observed in some South American economies (and even Israel) where inflation has lately been tamed. One may also argue that an analogous reasoning explains why in countries, like Turkey, where use of dollar is widespread in spot market transactions, so is the use of dollar-indexed debt.

The theory presented, strictly interpreted, demonstrates a reason why indexed bonds are not exchanged by private individuals. But in the case of trade in government bonds, at least at the point of issue, one of the parties to the trade is not a utility maximizing private agent. However, note our theory would apply just as well to any secondary trade of indexed government bonds between private individuals. Thus our theory predicts, this secondary market typically be a rather thin market. In turn, this carries the implication of indexed government debt not being a particularly liquid asset. Clearly, given rational, forward looking individuals, this would, in itself, ensure that demand for such debt would be weak even at the point of issue.

Finally, we turn briefly to some policy implications. One obviously welfare increasing move would be to publish (trustworthy) indexes that are more particular and focused on fewer goods and services than the CPI. Looking back to the formal model, if there were an index composed purely of the prices of x and z the resulting allocation would indeed be a Pareto improvement on the allocation obtained with index composed of prices of x, z and y (the market would become as good as complete for the h-type agents). Government action in introducing statutory indexation of some payments, say of wages in some sectors, would increase the trade in indexed debt. However, it is far from clear, given the abstractions in our model, that we may conclude that issuing such a fiat would at all be welfare enhancing.

6 Appendix

6.1 Slice-comonotonicity

The computation of the Choquet expectation operator using product capacities is particularly simple for *slice comonotonic* functions ((Ghirardato, 1997)), defined below. Let $X_1, ..., X_n$ be n (finite) sets and let $\Omega = X_1 \times ... \times X_n$. Correspondingly, let ν_i be convex non-additive probabilities defined on algebras of subsets of X_i, $i = 1, ..., n$.

Definition 2. *Let $\varphi : \Omega \to \mathbb{R}$. We say that φ has comonotonic x_i-sections if for every $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, $\left(x'_1, ..., x'_{i-1}, x'_{i+1}, ..., x'_n\right) \in X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n$, $\varphi(x_1, ..., x_{i-1}, \cdot, x_{i+1}, ..., x_n) : X_i \to \mathbb{R}$, and $\varphi\left(x'_1, ..., x'_{i-1}, \cdot, x'_{i+1}, ..., x'_n\right) : X_i \to \mathbb{R}$ are comonotonic functions. φ is called slice-comonotonic if it has comonotonic x_i-sections for every $i \in \{1, ..., n\}$.*

The utility function of agent h, $f_h(u(x_h^\omega, z_h^\omega))$, is actually slice comonotonic, since u is strictly monotone and hence Fact 1, below, which follows from Proposition 7 and Theorem 1 in (Ghirardato, 1997), applies to the calculation of Choquet expected utility for agent h.

Fact 1 *Suppose that $\varphi : \Omega \to \mathbb{R}$ is slice comonotonic. Then*

$$\int_\Omega \varphi(x_1, ..., x_n)\, d(\otimes \nu_i) = \int_{X_1} ... \int_{X_n} \varphi(x_1, ..., x_n)\, d\nu_n ... d\nu_1$$

6.2 A formal confirmation of the argument in Section 3

To see the argument put more formally, compute first the right-hand-side derivative of $V(x^1, x^2, x^3, x^4)$, in which we replace x^s by its expression as a function of b^i, to obtain a function $W(b^i)$:

$$\begin{aligned}
&\frac{dW(b^i)}{db^i} \mid_{\, b^i = 0^+} = \\
&\quad \mu \left[\nu^H \left(1 + \frac{p_y^H}{p_x} - \frac{q^i}{q^n p_x} \right) + (1 - \nu^H) \left(1 + \frac{p_y^L}{p_x} - \frac{q^i}{q^n p_x} \right) \right] \times \\
&\quad u'\left(\bar{x} + \frac{S}{q^n p_x} \right) + \\
&\quad (1 - \mu) \left[\nu^H \left(1 + \frac{p_y^H}{p_x} - \frac{q^i}{q^n \lambda p_x} \right) + (1 - \nu^H) \left(1 + \frac{p_y^L}{p_x} - \frac{q^i}{q^n \lambda p_x} \right) \right] \times \\
&\quad u'\left(\bar{x} + \frac{S}{q^n \lambda p_x} \right)
\end{aligned}$$

The same computation, but for the left-hand-side derivative yields:

$$\frac{dW(b^i)}{db^i} \mid_{b^i=0^-} =$$

$$\mu \left[(1-\nu^L) \left(1 + \frac{p_y^H}{p_x} - \frac{q^i}{q^n p_x} \right) + \nu^L \left(1 + \frac{p_y^L}{p_x} - \frac{q^i}{q^n p_x} \right) \right] \times$$
$$u' \left(\bar{x} + \frac{S}{q^n p_x} \right) +$$
$$(1-\mu) \left[(1-\nu^L) \left(1 + \frac{p_y^H}{p_x} - \frac{q^i}{q^n \lambda p_x} \right) + \nu^L \left(1 + \frac{p_y^L}{p_x} - \frac{q^i}{q^n \lambda p_x} \right) \right] \times$$
$$u' \left(\bar{x} + \frac{S}{q^n \lambda p_x} \right)$$

Hence, if the following conditions hold, then $b^i = 0$ is optimal for the agent:

$$\mu \left[1 - \tfrac{q^i}{q^n p_x} + \left(\nu^H \tfrac{p_y^H}{p_x} + (1-\nu^H) \tfrac{p_y^L}{p_x} \right) \right] u' \left(\bar{x} + \tfrac{S}{q^n p_x} \right) +$$
$$(1-\mu) \left[1 - \tfrac{q^i}{q^n \lambda p_x} + \left(\nu^H \tfrac{p_y^H}{p_x} + (1-\nu^H) \tfrac{p_y^L}{p_x} \right) \right] u' \left(\bar{x} + \tfrac{S}{q^n \lambda p_x} \right)$$
$$\leq 0 \leq$$
$$\mu \left[1 - \tfrac{q^i}{q^n p_x} + \left((1-\nu^L) \tfrac{p_y^H}{p_x} + \nu^L \tfrac{p_y^L}{p_x} \right) \right] u' \left(\bar{x} + \tfrac{S}{q^n p_x} \right) +$$
$$(1-\mu) \left[1 - \tfrac{q^i}{q^n \lambda p_x} + \left((1-\nu^L) \tfrac{p_y^H}{p_x} + \nu^L \tfrac{p_y^L}{p_x} \right) \right] u' \left(\bar{x} + \tfrac{S}{q^n \lambda p_x} \right)$$

This amounts to:

$$\nu^H \frac{p_y^H}{p_x} + (1-\nu^H)\frac{p_y^L}{p_x} \leq$$
$$\frac{\mu \frac{q^i}{q^n p_x} u'\left(\bar{x}+\frac{S}{q^n p_x}\right)+(1-\mu)\frac{q^i}{q^n \lambda p_x} u'\left(\bar{x}+\frac{S}{q^n \lambda p_x}\right)}{\mu u'\left(\bar{x}+\frac{S}{q^n p_x}\right)+(1-\mu) u'\left(\bar{x}+\frac{S}{q^n \lambda p_x}\right)} - 1$$
$$\leq (1-\nu^L)\frac{p_y^H}{p_x} + \nu^L \frac{p_y^L}{p_x}$$

If one looks at the limiting case in which $\lambda = 1$, this further reduces to:

$$\nu^H p_y^H + (1-\nu^H)p_y^L + p_x \leq \frac{q^i}{q^n} \leq (1-\nu^L)p_y^H + \nu^L p_y^L + p_x$$

where it is easily seen that there is a range of relative prices for the indexed bond with respect to the nominal bond such that it is not held in the optimal portfolio, as long as $\nu^L + \nu^H < 1$. If, on the other hand, the capacity ν is actually a probability measure $\nu^L + \nu^H = 1$, there is only one relative price q^i/q^n such that the agent does not want to hold any indexed bond. By continuity, the same is true when λ is strictly greater than 1. Observe that the length of the interval at which the agent does not want to hold any position in the indexed bond is increasing in the ambiguity of the beliefs, measured by $1 - \nu^L - \nu^H$.

6.3 Proofs of results in Section 4

Proof of Proposition 1.

Proof. Since $\frac{p_x^1}{p_z^1} = \frac{p_x^3}{p_z^3}$, there exists β such that $(p_x^1, p_z^1) = \frac{1}{\beta}(p_x^3, p_z^3)$. We want to show that $\beta = 1$. From the equilibrium condition on the money market and the definition of the states (which entails that $M^1 = M^3$), we have that:

$$p_x^1 \sum_{h=1}^{H} \bar{x}_h + p_y^1 y^L + p_z^1 \left(\sum_{h=1}^{H} \bar{z}_h + \sum_{k=1}^{K} \bar{z}_k \right)$$
$$= p_x^3 \sum_{h=1}^{H} \bar{x}_h + p_y^3 y^H + p_z^3 \left(\sum_{h=1}^{H} \bar{z}_h + \sum_{k=1}^{K} \bar{z}_k \right)$$

Replacing p_x^3 and p_z^3 by βp_x^1 and βp_z^1 respectively, in the equation above, and recalling that, at equilibrium,

$$\frac{p_y^1}{p_z^1} = \frac{p_y^3}{p_z^3}\frac{y^H}{y^L}$$

and hence,

$$\frac{p_y^3}{p_z^1} = \beta \frac{p_y^1}{p_z^1}\frac{y^L}{y^H}$$

one gets:

$$\frac{p_x^1}{p_z^1}(1-\beta)\sum_{h=1}^{H}\bar{x}_h + \frac{p_y^1}{p_z^1}(1-\beta)y^L + (1-\beta)\left(\sum_{h=1}^{H}\bar{z}_h + \sum_{k=1}^{K}\bar{z}_k\right) = 0$$

Hence, $\beta = 1$. The same reasoning shows that $p_x^2 = p_x^4$ and $p_z^2 = p_z^4$.

Proof of Proposition 2

Proof. Recall that $\frac{p_x^1}{p_z^1} = \zeta = \frac{p_x^2}{p_z^2}$ and $\frac{p_y^1}{p_z^1} = \frac{p_y^2}{p_z^2}$. Hence, (p_x^1, p_y^1, p_z^1) and (p_x^2, p_y^2, p_z^2) are proportional. Furthermore,

$$p_x^1\sum_{h=1}^{H}\bar{x}_h + p_y^1 y^L + p_z^1\left(\sum_{h=1}^{H}\bar{z}_h + \sum_{k=1}^{K}\bar{z}_k\right) = m$$

and

$$p_x^2\sum_{h=1}^{H}\bar{x}_h + p_y^2 y^L + p_z^2\left(\sum_{h=1}^{H}\bar{z}_h + \sum_{k=1}^{K}\bar{z}_k\right) = M$$

and hence

$$(p_x^1, p_y^1, p_z^1) = \frac{1}{\lambda}(p_x^2, p_y^2, p_z^2)$$

An analogous argument holds for states 3 and 4.

Notation 1 *To simplify notation, the remaining proofs apply the terms $\alpha_h^\omega(b_h^i, b_h^n)$ and $\xi_h^\omega(b_h^i, b_h^n)$ (that we will write as α_h^ω and ξ_h^ω, respectively, for the sake of simplicity) defined as follows:*

$$\alpha_h^\omega(b_h^i, b_h^n) \equiv \frac{p_x^\omega\bar{x}_h + p_z^\omega\bar{z}_h + (p_x^\omega + p_y^\omega + p_z^\omega)b_h^i + b_h^n + \bar{m}_h}{p_x^\omega\sum_{h=1}^{H}\bar{x}_h + p_z^\omega\sum_{h=1}^{H}\bar{z}_h},$$

$$\xi_h^\omega(b_h^i, b_h^n) = \frac{f_h'\left(\alpha_h^\omega(b_h^i, b_h^n)u\right)}{p_x^\omega\sum_{h=1}^{H}\bar{x}_h + p_z^\omega\sum_{h=1}^{H}\bar{z}_h}u.$$

Lemma 1. *At an equilibrium, if $b_h^i > 0$, then $\alpha_h^1 > \alpha_h^3$ and $\alpha_h^2 > \alpha_h^4$. If $b_h^i < 0$, then $\alpha_h^1 < \alpha_h^3$ and $\alpha_h^2 < \alpha_h^4$.*

Proof. By definition of α_h^ω and since $p_x^1 = p_x^3$ and $p_z^1 = p_z^3$, and the endowment in x and z are non random, one has:

$$\alpha_h^1 - \alpha_h^3 = \frac{(p_y^1 - p_y^3)b_h^i}{p_x^1\sum_{h=1}^{H}\bar{x}_h + p_z^1\sum_{h=1}^{H}\bar{z}_h}$$

Hence, the sign of $\alpha_h^1 - \alpha_h^3$ is the same as the sign of b_h^i since $p_y^1 > p_y^3$ (recall that $p_y^1/p_y^3 = y^H/y^L > 1$).

Proof of Theorem 1

Proof. (Sketch.) Since the hypothesis of this theorem is that $\mu^m + \mu^M = 1$ and $\nu^H + \nu^L = 1$, we may write the proof by setting $\mu^m \equiv \mu$, $\mu^M \equiv 1 - \mu$ and $\nu^H = \nu$, $\nu^L = 1 - \nu$.

Write down the f.o.c. to agent h's program:

$$\begin{aligned}&-q^i u^0 + \mu(1-\nu)\xi_h^1(b_h^i, b_h^n)(p_x^1 + p_y^1 + p_z^1) + \\ &(1-\mu)(1-\nu)\xi_h^2(b_h^i, b_h^n)(p_x^2 + p_y^2 + p_z^2) \\ &+\mu\nu\xi_h^3(b_h^i, b_h^n)(p_x^3 + p_y^3 + p_z^3) + \\ &(1-\mu)\nu\xi_h^4(b_h^i, b_h^n)(p_x^4 + p_y^4 + p_z^4) = 0\end{aligned}$$

$$\begin{aligned}&-q^n u^0 + \mu(1-\nu)\xi_h^1(b_h^i, b_h^n) + (1-\mu)(1-\nu)\xi_h^2(b_h^i, b_h^n) \\ &+\mu\nu\xi_h^3(b_h^i, b_h^n) + (1-\mu)\nu\xi_h^4(b_h^i, b_h^n) = 0\end{aligned}$$

Recall that $(p_x^2, p_y^2, p_z^2) = \lambda(p_x^1, p_y^1, p_z^1)$, $(p_x^3, p_y^3, p_z^3) = \lambda(p_x^4, p_y^4, p_z^4)$, $p_x^1 = p_x^3$, $p_z^1 = p_z^3$, $p_x^2 = p_x^4$, $p_z^2 = p_z^4$, and $p_y^1/p_y^3 = y^H/y^L = p_y^2/p_y^4$.

Observe that $\xi_h^1(0, b_h^n) = \xi_h^3(0, b_h^n)$ and $\xi_h^2(0, b_h^n) = \xi_h^4(0, b_h^n)$. Hence, at an equilibrium, if $b_h^i = 0$ for all h, it must be the case that:

$$\begin{aligned}&-q^i u^0 + \mu(1-\nu)\xi_h^1(0, b_h^n)(p_x^1 + p_y^1 + p_z^1) + \\ &(1-\mu)(1-\nu)\xi_h^2(0, b_h^n)\lambda(p_x^1 + p_y^1 + p_z^1) \\ &+\mu\nu\xi_h^1(0, b_h^n)(p_x^1 + p_z^1 + p_y^1 y^L/y^H) + \\ &(1-\mu)\nu\xi_h^2(0, b_h^n)\lambda(p_x^2 + p_z^2 + p_y^2 y^L/y^H) = 0\end{aligned}$$

$$-q^n u^0 + \mu\xi_h^1(0, b_h^n) + (1-\mu)\xi_h^2(0, b_h^n) = 0$$

that is

$$\begin{aligned}&-q^i u^0 + [p_x^1 + p_z^1 + p_y^1(1 - \nu + \nu y^L/y^H)][\mu\xi_h^1(0, b_h^n) + (1-\mu)\lambda\xi_h^2(0, b_h^n)] = 0 \\ &-q^n u^0 + \mu\xi_h^1(0, b_h^n) + (1-\mu)\xi_h^2(0, b_h^n) = 0\end{aligned}$$

One can see straight away on these two equations that, unless $\lambda = 1$, there is "little chance" that there exists (q^i, q^n) and $(b_h^n)_{h=1,\dots,H}$ such that they hold for all h. The following argument sketches how we might put this formally. We define a mapping ϕ from $\mathbb{R}^{H+3}$ to $\mathbb{R}^{2H+1}$ as follows:

$$\phi(b_1^n, \dots, b_H^n, q^i, q^n, u^0) =$$

$$\begin{bmatrix} \vdots \\ \mu\xi_h^1(0, b_h^n) + (1-\mu)\lambda\xi_h^2(0, b_h^n) - \frac{q^i u^0}{p_x^1 + p_z^1 + p_y^1(1-\nu+\nu y^L/y^H)} \\ \mu\xi_h^1(0, b_h^n) + (1-\mu)\lambda\xi_h^2(0, b_h^n) - q^n u^0 \\ \vdots \\ \sum_{h=1}^H b_h^n \end{bmatrix}$$

Observe, the equilibria with no trade in the indexed bond are the zeros of this function. The argument then runs as follows. Note that the Jacobian of ϕ, a

$(2H+1)\times(H+3)$ matrix, has full rank. Hence, by a transversality argument, we may conclude that for generic u^0, the rank of the matrix is full. Now, this implies that the system does not have a solution, since "there are more equations than unknowns". Hence, $b_h^i = 0$ for all h cannot be an equilibrium for generic first period aggregate endowments (the latter is sufficient to change u^0).

Proof of Theorem 2(a)

Proof. Now, to write down the Choquet integral w.r.t. an agent's portfolio, one has to consider all possible cases (drop subscript h), whether $b^i > 0$ or $b^i < 0$ and whether $b^n + \bar{m} > 0$ or $b^n + \bar{m} < 0$. Observe that:

$$\begin{array}{ll} b^i > 0 & \Rightarrow \alpha^1 > \alpha^3 \ \alpha^2 > \alpha^4 \\ b^i < 0 & \Rightarrow \alpha^1 < \alpha^3 \ \alpha^2 < \alpha^4 \\ b^n + \bar{m} > 0 & \Rightarrow \alpha^1 > \alpha^2 \ \alpha^3 > \alpha^4 \\ b^n + \bar{m} < 0 & \Rightarrow \alpha^1 < \alpha^2 \ \alpha^3 < \alpha^4 \end{array}$$

For each case, there are two possible orders, but these two orders give the same (probabilistic) "decision weights". For instance, if $b^i > 0$ and $b^n + \bar{m} > 0$, the two following orders are possible: $\alpha^1 > \alpha^2 > \alpha^3 > \alpha^4$ or $\alpha^1 > \alpha^3 > \alpha^2 > \alpha^4$. If one computes the Choquet integral in these two cases, one sees that they take the same form, *i.e.* the switch in the "middle position" has no effect. Decision weights associated to the different cases:

	$\omega = 1$	$\omega = 2$
$b^i > 0, b^n + \bar{m} > 0$	$\nu^L \mu^m$	$\nu^L(1-\mu^m)$
$b^i > 0, b^n + \bar{m} < 0$	$\nu^L(1-\mu^M)$	$\nu^L \mu^M$
$b^i < 0, b^n + \bar{m} > 0$	$(1-\nu^H)\mu^m$	$(1-\nu^H)(1-\mu^m)$
$b^i < 0, b^n + \bar{m} < 0$	$(1-\nu^H)(1-\mu^M)$	$(1-\nu^H)\mu^M$

	$\omega = 3$	$\omega = 4$
$b^i > 0, b^n + \bar{m} > 0$	$(1-\nu^L)\mu^m$	$(1-\nu^L)(1-\mu^m)$
$b^i > 0, b^n + \bar{m} < 0$	$(1-\nu^L)(1-\mu^M)$	$(1-\nu^L)\mu^M$
$b^i < 0, b^n + \bar{m} > 0$	$\nu^H \mu^m$	$\nu^H(1-\mu^m)$
$b^i < 0, b^n + \bar{m} < 0$	$\nu^H(1-\mu^M)$	$\nu^H \mu^M$

Suppose first that $b_h^i > 0$. One gets, from the first order conditions

$$\frac{q^i}{q^n} = \frac{\pi\nu^L\xi_h^1(p_x^1+p_y^1+p_z^1)+(1-\pi)\nu^L\xi_h^2(p_x^2+p_y^2+p_z^2)}{\pi\nu^L\xi_h^1+(1-\pi)\nu^L\xi_h^2+\pi(1-\nu^L)\xi_h^3+(1-\pi)(1-\nu^L)\xi_h^4} + \frac{\pi(1-\nu^L)\xi_h^3(p_x^3+p_y^3+p_z^3)+(1-\pi)(1-\nu^L)\xi_h^4(p_x^4+p_y^4+p_z^4)}{\pi\nu^L\xi_h^1+(1-\pi)\nu^L\xi_h^2+\pi(1-\nu^L)\xi_h^3+(1-\pi)(1-\nu^L)\xi_h^4}$$

where π depends on the agent's position on the nominal bond market: if $b_h^n + \bar{m}_h > 0$, $\pi = \mu^m$, if $b_h^n + \bar{m}_h < 0$, $\pi = (1-\mu^M)$ and, lastly, if $b_h^n + \bar{m}_h = 0$, π is some number lying in the interval $\left[\mu^m, 1-\mu^M\right]$.

Observe that $(\alpha_h^1(b_h^i, b_h^n), \alpha_h^3(b_h^i, b_h^n))$ and (p_y^1, p_y^3) are positively dependent (see chapter 3 in (Magill and Quinzii, 1996)). Hence, since f_h is concave,

$$cov\left(\left(\alpha_h^1\left(b_h^i, b_h^n\right), \alpha_h^3\left(b_h^i, b_h^n\right)\right), \left(p_y^1, p_y^3\right)\right) < 0$$

Similarly,

$$cov\left(\left(\alpha_h^2\left(b_h^i, b_h^n\right), \alpha_h^4\left(b_h^i, b_h^n\right)\right), \left(p_y^2, p_y^4\right)\right) < 0$$

Hence, a necessary condition for having $b_h^i > 0$ at an equilibrium is that:

$$\frac{q^i}{q^n} < \frac{\pi\left(\nu^L\xi_h^1+(1-\nu^L)\xi_h^3\right)\left(\nu^L(p_x^1+p_y^1+p_z^1)+(1-\nu^L)\pi(p_x^3+p_y^3+p_z^3)\right)}{\pi\left(\nu^L\xi_h^1+(1-\nu^L)\xi_h^3\right)+(1-\pi)\left(\nu^L\xi_h^2+(1-\nu^L)\xi_h^4\right)} + \frac{(1-\pi)\left(\nu^L\xi_h^2+(1-\nu^L)\xi_h^4\right)\left(\nu^L(p_x^2+p_y^2+p_z^2)+(1-\nu^L)(p_x^4+p_y^4+p_z^4)\right)}{\pi\left(\nu^L\xi_h^1+(1-\nu^L)\xi_h^3\right)+(1-\pi)\left(\nu^L\xi_h^2+(1-\nu^L)\xi_h^4\right)}$$

and therefore

$$\frac{q^i}{q^n} < \max\left(\nu^L(p_x^1+p_y^1+p_z^1)+(1-\nu^L)(p_x^3+p_y^3+p_z^3),\ \nu^L(p_x^2+p_y^2+p_z^2)+(1-\nu^L)(p_x^4+p_y^4+p_z^4)\right)$$

Recalling that

$$p_x^3 = p_x^1, \quad p_y^3 = \tfrac{y^L}{y^H}p_y^1, \quad \text{and } p_z^3 = p_z^1$$
$$p_x^4 = p_x^2, \quad p_y^4 = \tfrac{y^L}{y^H}p_y^2, \quad \text{and } p_z^4 = p_z^2$$

one gets that a necessary condition for $b_h^i > 0$ at equilibrium is that:

$$\frac{q^i}{q^n} < \max\begin{pmatrix} p_x^1 + \left(\frac{y^L}{y^H} + \nu^L\left(1-\frac{y^L}{y^H}\right)\right)p_y^1 + p_z^1, \\ p_x^2 + \left(\frac{y^L}{y^H} + \nu^L\left(1-\frac{y^L}{y^H}\right)\right)p_y^2 + p_z^2 \end{pmatrix}$$

Furthermore, since $(p_x^2, p_y^2, p_z^2) = \lambda(p_x^1, p_y^1, p_z^1)$, a sufficient condition for agent h to hold a positive position in the indexed bond at equilibrium is:

$$\frac{q^i}{q^n} < \lambda\left(p_x^1 + \left(\frac{y^L}{y^H} + \nu^L\left(1-\frac{y^L}{y^H}\right)\right)p_y^1 + p_z^1\right)$$

Observe that this condition is independent of the position of the agent on the nominal bond market, as the weights π, which depend on the sign of $b_h^n + \bar{m}_h$, do not show up in this condition.

Consider now an agent who, at an equilibrium, holds a negative amount of the indexed bond, that is $b_h^i < 0$. One gets, from the first order conditions

$$\frac{q^i}{q^n} = \frac{\pi(1-\nu^H)\xi_h^1(p_x^1+p_y^1+p_z^1)+(1-\pi)(1-\nu^H)\xi_h^2(p_x^2+p_y^2+p_z^2)}{\pi(1-\nu^H)\xi_h^1+(1-\pi)(1-\nu^H)\xi_h^2+\pi\nu^H\xi_h^3+(1-\pi)\nu^H\xi_h^4} + \frac{\pi\nu^H\xi_h^3(p_x^3+p_y^3+p_z^3)+(1-\pi)\nu^H\xi_h^4(p_x^4+p_y^4+p_z^4)}{\pi(1-\nu^H)\xi_h^1+(1-\pi)(1-\nu^H)\xi_h^2+\pi\nu^H\xi_h^3+(1-\pi)\nu^H\xi_h^4}$$

where, as above, the value of π depends on the sign of $b_h^n + \bar{m}_h$ for agent h. Note that given that we look at one particular agent, the π that appears here is the same as the one that appeared in the f.o.c. for agent h if he had a positive position in the indexed bond ($b_h^i > 0$).

Noticing that $\left(\alpha_h^1(b_h^i, b_h^n), \alpha_h^3(b_h^i, b_h^n)\right)$ and $\left(p_y^1, p_y^3\right)$ are now negatively dependent, and hence

$$cov\left(\left(\alpha_h^1\left(b_h^i, b_h^n\right), \alpha_h^3\left(b_h^i, b_h^n\right)\right), \left(p_y^1, p_y^3\right)\right) > 0$$

as well as

$$cov\left(\left(\alpha_h^2\left(b_h^i, b_h^n\right), \alpha_h^4(b_h^i, b_h^n)\right), \left(p_y^2, p_y^4\right)\right) > 0$$

one gets the following necessary condition for an agent to go short on the indexed bond:

$$\frac{q^i}{q^n} > p_x^1 + \left(\frac{y^L}{y^H} + (1 - \nu^H)\left(1 - \frac{y^L}{y^H}\right)\right) p_y^1 + p_z^1$$

Here also, the exact value of π does not matter given that it does not appear in the expression.

Therefore, if

$$\lambda\left(p_x^1 + \left(\frac{y^L}{y^H} + \nu^L\left(1 - \frac{y^L}{y^H}\right)\right) p_y^1 + p_z^1\right) \tag{1}$$
$$< p_x^1 + \left(\frac{y^L}{y^H} + (1 - \nu^H)\left(1 - \frac{y^L}{y^H}\right)\right) p_y^1 + p_z^1$$

then, at an equilibrium, no agent wishes to trade the indexed bond, and therefore, $b_h^i = 0$ for all h. This condition can be expressed as follows, writing $1 - \nu^H = \mathcal{A}(\nu) + \nu^L$:

$$\lambda < \frac{p_x^1 + \left(\frac{y^L}{y^H} + \left(\mathcal{A}(\nu) + \nu^L\right)\left(1 - \frac{y^L}{y^H}\right)\right) p_y^1 + p_z^1}{p_x^1 + \left(\frac{y^L}{y^H} + \nu^L\left(1 - \frac{y^L}{y^H}\right)\right) p_y^1 + p_z^1}$$

Let

$$\delta = \frac{p_x^1 + p_y^1 + p_z^1}{p_x^1 + \frac{y^L}{y^H} p_y^1 + p_z^1}$$

corresponding to $\mathcal{A}(\nu) = 1$ and $\nu^L = \nu^H = 0$. Notice that $\delta > 1$. Furthermore, if $\lambda < \delta$, there exists $\gamma < 1$ such that if $\mathcal{A}(\nu) > \gamma$, then the condition (1) is met, and, therefore, at an equilibrium, $b_h^i = 0$ for all h. Notice that δ can be expressed as a function of the fundamentals of the model, since, say p_x^1/p_z^1 and p_y^1/p_z^1 are known as functions of the aggregate endowments and the utility function u.

Proof of Theorem 2(b)

Proof. We show that if $(b_h^i, b_h^n) = (0,0)$ for all h is an equilibrium of the model when beliefs about the money supply are represented by the capacity μ, then, perturbing slightly μ, one gets a new equilibrium at which $b_h^n \neq 0$ and $b_{h'}^n \neq 0$, for some h, h'.Note first that, if $\bar{m}_h > 0$, $b_h^n = 0$ implies that $\alpha^1 > \alpha^2$ and $\alpha^3 > \alpha^4$, while if $\bar{m}_{h'} < 0$, $b_{h'}^n = 0$ implies that $\alpha^1 < \alpha^2$ and $\alpha^3 < \alpha^4$.

Now, the first order condition with respect to b_h^n, taken at $b_h^n = 0$ is:

$$\begin{aligned} q^n u^0 = &\pi \mu^m \xi_h^1(b_h^i, 0) + \pi(1-\mu^m)\xi_h^2(b_h^i, 0) \\ &+ (1-\pi)\mu^m \xi_h^3(b_h^i, 0) + (1-\pi)(1-\mu^m)\xi_h^4(b_h^i, 0) \end{aligned}$$

where π is some decision weight related to the capacity ν that we need not specify at this stage. Indeed, observe that $\xi_h^1(0,0) = \xi_h^3(0,0)$ and $\xi_h^2(0,0) = \xi_h^4(0,0)$. Hence, given that $\bar{m}_h > 0$, a necessary condition for $(b_h^i, b_h^n) = (0,0)$ to be a solution to h's maximization program is simply that:

$$q^n u^0 = \mu^m \xi_h^1(0,0) + (1-\mu^m)\xi_h^2(0,0)$$

Similarly, a necessary condition for $(b_{h'}^i, b_{h'}^n) = (0,0)$ to be a solution to h''s maximization program is that:

$$q^n u^0 = (1-\mu^M)\xi_{h'}^1(0,0) + \mu^M \xi_{h'}^2(0,0)$$

Therefore, a necessary condition for $(b_h^i, b_h^n) = (0,0)$, h, h', at an equilibrium is that

$$\mu^m \xi_h^1(0,0) + (1-\mu^m)\xi_h^2(0,0) = (1-\mu^M)\xi_{h'}^1(0,0) + \mu^M \xi_{h'}^2(0,0)$$

Now, either this condition does not hold, and then $(0,0)$ is not an equilibrium of the economy and therefore, since we know there is no trade in the indexed bond, this means that there is some trade in the nominal bond. Or else, this condition does hold. However, this essentially means that

$$\mu^m = \frac{(1-\mu^M)\xi_{h'}^1(0,0) + \mu^M \xi_{h'}^2(0,0) - \xi_h^2(0,0)}{\xi_h^1(0,0) - \xi_h^2(0,0)}$$

Hence, if this were the case, that property would not hold for any other economy in which we would have changed μ^m ever so slightly.

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Sunspots, Indeterminacy and Pareto Inefficiency in Economies With Incomplete Markets*

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Summary. We consider two periods economies with both intrinsic and extrinsic uncertainty. Asset markets are incomplete in the certainty economy. If assets are nominal, there are enough commodities and the number of agents is greater than two and smaller than the total number of states of nature "tomorrow" (minus one), then a sunspot-invariant equilibrium is generically Pareto dominated by some sunspot equilibria. When assets are real, and there are enough commodities, whenever there are sunspot equilibria, typically there are also sunspot equilibria Pareto dominating the sunspot-invariant equilibria under the same restriction on the number of agents (and stronger restrictions on the number of commodities).

1 Introduction

Cass [3] provided the first example of the existence of a continuum of sunspot equilibria in economies with incomplete markets. Since then, existence and structure of sunspot equilibria have been extensively studied in this class of economies with nominal (see, [18], [4], [14], [16], [20] and [10]) and with real assets (see, [13] and [9])). Existence and "number" of sunspot equilibria in financial economies crucially depend upon the nature of asset payoffs: With nominal assets, they generically exist in a neighborhood of a sunspot-invariant equilibrium and their set contains a non-zero dimensional manifold. Its dimension depends upon the way asset prices are treated: Generically, it is $(\Sigma - I)$ if asset prices are fixed, $(\Sigma - 1)$ if they are variable (Σ is the number of states of nature "tomorrow", I the number of assets). When asset payoffs are real (and if there are enough commodities), sunspot equilibria exist for an open set of asset structures and are locally unique.

Our second theme is Pareto dominance. As pointed out in [6], in convex economies sunspot equilibria are Pareto inefficient. When asset markets are

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incomplete at the sunspot-invariant equilibria, they are themselves typically Pareto inefficient and can possibly be Pareto dominated by equilibria of the same economy where extrinsic uncertainty matters. Parametric examples of this phenomenon in incomplete market economies have been provided in the literature ([15] gives an example with $\Sigma = 3$, two agents and one commodity).

Our purpose is to investigate the (comparative) efficiency properties of sunspot and sunspot-invariant equilibria. When asset payoffs are nominal, we show that, generically, sunspot-invariant equilibria are Pareto dominated by some sunspot equilibria, if the number of agents (H) is greater than two and smaller than $(\Sigma - 1)$ and there are enough commodities. We focus the analysis on this comparison. However, an obvious corollary is that, typically, sunspot equilibria are in turn dominated by some other equilibria. A second corollary is that, when $2 < H < (S - 1)$ (S is the number of intrinsic events "tomorrow"), certainty equilibria are typically Pareto dominated by some other certainty equilibria.

The notion of genericity adopted is somewhat weak, because we show that, given a vector of numeraire, there is an open, dense set of economies (parameterized by utility functions and endowments) such that our result holds. A stronger result would be to show that, generically, all the equilibria in a full measure set of sunspot invariant equilibria are Pareto dominated by some sunspot equilibrium. I am not ready to formulate a conjecture on the truthfulness of this result, which could possibly be analyzed using the approach exploited in [17] to study real indeterminacy from a global viewpoint in economies with nominal assets and incomplete markets.

Given [11] (where real indeterminacy of equilibria is re-interpreted in terms of real effects of monetary policies), one possible interpretation of our result is in terms of the existence of Pareto improving randomized monetary policies. Another possible interpretation is as another robust example of a class of economies where the value of information can be negative.

We also study the case of economies with real assets, restricting ourselves to financial structures such that sunspot equilibria exist and with enough commodities. We show that, for a generic set of economies (parameterized by utility functions and endowments), there are asset structures such that sunspot equilibria exist and Pareto dominate at least one of the sunspot-invariant equilibria.

The structure of the paper is the following: Section 2 describes the model. Section 3 establishes that, with nominal assets, sunspot-invariant equilibria are, generically, regular in the sunspot economy and (if they are not Pareto efficient) satisfy some further restrictions on the vector of excess demand and Lagrange multipliers. Then, following the approach outlined in [19] and developed in [8], [5] and [7], we establish that the *extended system of equations* describing the equilibria of the economy and the utility functions of the agents define a system of independent equations (i.e., the derivative of this transformation has maximal rank, equal to the number of equations). This immediately implies that, given a sunspot-invariant equilibrium, there are open sets of Pareto superior sunspot equilibria. While the logic of our argument is the same as the one of the quoted papers, the technical details are quite different. Those papers establish the full rank property exploiting in a crucial way arbitrary perturbations of the second order derivatives of the utility functions. This approach works in our framework when $(S-1) \geq H$, it is bound to fail when $H > (S-1)$, i.e., in the more interesting case when "sunspot matters" in terms of feasibility of Pareto improvements. Therefore, we have to follow a more cumbersome approach.

The last section studies economies with real assets, building on the previous results.

2 The model

We consider a two period model with both intrinsic and extrinsic uncertainty. Spot $s = 0$ is "today", $s = 1, ..., S$ denotes one of the S intrinsic events "tomorrow". There are also K mutually exclusive extrinsic events, so that, in the sunspot economy, there are $KS = \Sigma$ states "tomorrow". A generic spot will be denoted either by σ, $\sigma = 0, (11), ..., (1S), ..., (K1), ..., (KS)$, or by the more explicit notation ks, when it is important to emphasize the *intrinsic* event characterizing the state of nature.

To simplify notation, and without any loss of generality, we assume that the probability of each extrinsic event is $1/K$.

There are H agents, denoted by $h = 1, ..., H$. At each spot there are C commodities, $c = 1, ..., C$, so that the total number of commodities is $G = ((\Sigma + 1)C)$. Agent h' consumption vector at spot σ is $x_h^\sigma = (x_h^{\sigma 1}, ..., x_h^{\sigma C})$, while his consumption vector is $x_h = (x_h^0, ..., x_h^\Sigma)$. In a similar fashion, excess demand vectors are z_h and z_h^σ, while commodity prices are denoted by $p = (p^0, ..., p^\Sigma)$ and $p^\sigma = (p^{\sigma 1}, ..., p^{\sigma C})$. Also, let $\Psi(p)$ be the $((\Sigma + 1) \times G)$ dimensional matrix

$$\Psi(p) = \begin{bmatrix} p^0 & & \\ & \ddots & \\ & & p^\Sigma \end{bmatrix}.$$

There are I assets. Asset i has payoffs $y^i = [y^{1i}, ..., y^{\Sigma i}]$, its price is q^i. Let Y be the $(\Sigma \times I)$ dimensional matrix of asset payoffs and let

$$R(q) = \begin{bmatrix} -q \\ Y \end{bmatrix}$$

be the price-payoffs matrix. For each h, $b_h \in \Re^I$ is the portfolio vector. The payoffs of asset i can be nominal or can be given, at each state, by the value of a commodity bundle $\rho^{si} = [\rho^{si1}, ..., \rho^{siC}]$.

When assets are nominal, we need three restrictions on asset payoffs:

i. $y^{ks} = y^{k's}$, for each s and $k, k\prime$;

ii. The matrix $\left[y^{k1} \cdots y^{kS} \right]$ is in general position;
iii. $y^{\sigma 1} = 1$, for each $\sigma > 0$, i.e., asset one is inside money.

Evidently, i guarantees the existence of k-invariant equilibria, iii could be weakened.

For the case of real assets, we will explicitly construct the k-invariant collection ρ^{si}, for each s and i, in Section 4.

Assumptions on consumers are standard: For each h, the endowment vector is $e_h = (e_h^0, e_h^{11}, ..., e_h^{KS}) \in \Re^G_{++}$, satisfying $e_h^{ks} = e_h^{k's}$, for each s, k and $k\prime$. Given the k-invariance restriction, we will identify the endowment space with $\Re^{(S+1)C}_{++}$. Preferences are described by a Von Neumann-Morgenstern, strictly concave, strictly increasing, C^2, utility function

$$V_h(x_h) = \sum_k u_h(x_h^0, x_h^{k1}, ..., x_h^{kS})/K$$

such that the Bernoulli utility index $u_h(x_h^0,\ x_h^1,\ ...,\ x_h^S)$ satisfies the usual boundary restriction: the set $\{(x_h^0, x_h^1, ..., x_h^S) \in \Re^{(S+1)C}_{++} |\ u_h(x_h^0, x_h^1, ..., x_h^S) \geq u_h(x_h^{0*}, x_h^{1*}, ..., x_h^{S*})\}$ is closed in $\Re^{(S+1)C}_{++}$, for each $(x_h^{0*}, x_h^{1*}, ..., x_h^{S*}) \in \Re^{(S+1)C}_{++}$.

In the sequel, we will take as given asset payoffs Y (or ρ) and probabilities, π. The space of economies is identified with $\mathcal{E} = \prod_h (\mathcal{U}_h \times \Re^{(S+1)C}_{++})$, where $\mathcal{U}_h$ is the set of Bernoulli utility indexes satisfying the conditions above. As usual, we endow $\mathcal{U}_h$ with the C^2, compact-open topology, $\Re^{(S+1)C}_{++}$ with the Euclidean topology and $\mathcal{E}$ with the product topology. E will denote an element of $\mathcal{E}$.

When dealing with nominal assets, we need an appropriate parameterization of the set of equilibria. Here, the most convenient is the following: Fix $p^{\sigma C} = 1$, for each σ, and (locally) parameterize the set of equilibria with a vector $\nu = (1, \nu^{11}, \nu^{12}, ..., \nu^{\Sigma - 1}, 1)$, where we identify ν with its variable coefficients, so that $\nu \in \Re^{\Sigma-1}_{++}$. Then, the matrix $\Psi(p)$ is replaced by

$$\Psi(p, \nu) = \begin{bmatrix} p^0 & & & \\ & v^{11}p^{11} & & \\ & & \ddots & \\ & & & p^{\Sigma} \end{bmatrix}.$$

In the sequel, $p \in \mathbf{P} = \{p \in \Re^G_{++} | p^{\sigma C} = 1\}$ and $\nu \in \Re^{\Sigma-1}_{++}$. Also, let $G^{\setminus} = ((\Sigma + 1)(C - 1))$ and let $p^{\setminus} \in \Re^{G^{\setminus}}_{++}$ be the vector of the first $(C - 1)$ commodity prices at each spot. Finally, let's restrict q to the set of noarbitrage asset prices, $\mathbf{Q} = \left\{ q \in \Re^I |\ \theta R(q) = 0, \text{ for some } \theta \in \Re^{\Sigma+1}_{++} \right\}$.

Consumer behavior and equilibrium are defined as usual:

Definition 1. *Given $\nu \in \Re^{\Sigma-1}_{++}$, an equilibrium is a pair $(..., (x_h, b_h), ...)$ and $(p, q) \in \mathbf{P} \times \mathbf{Q}$, such that:*

i. For each h, $(x_h, b_h) \in \arg\max V_h(x_h)$ subject to $\Psi(p, \nu) z_h = R(q) b_h$;
ii. $\sum_h x_h = \sum_h e_h$ and $\sum_h b_h = 0$;

iii. An equilibrium is a sunspot equilibrium if, for some h and some s, there are k and k' such that $x_h^{ks} \neq x_h^{k's}$. *Otherwise, it is a k-invariant equilibrium.*

Bear in mind that, to avoid too many repetitions, we use the term sunspot equilibrium only to refer to equilibria where sunspots matter. Equilibria of the sunspot economy where sunspots do not matter are called k-invariant equilibria. Finally, we use the term certainty equilibria to refer to the equilibria of the certainty economy, i.e., (with some abuse of language) of the economy with no *extrinsic* uncertainty.

As is well known, k-invariant equilibria are related in an obvious way to certainty equilibria. In particular, $((p^{0*}, ..., p^{S*}),\ q^*) \in \Re^{(S+1)C+I}$ is a certainty equilibrium associated with $\mu^* \in \Re_{++}^{S-1}$, and with allocation $(...,(x_h^*, b_h^*), ...)$ and Lagrange multipliers $\lambda_h^* \in \Re_{++}^{S+1}$, for each h, if and only if $((p^{0*}, ..., (p^{1*}, ..., p^{S*}), ...), q^*) \in \Re^{G+I}$ is a k-invariant equilibrium associated with $\nu = (1, \mu^*, ..., \mu^*)$, allocation $(..., ((x_h^{0*}, ..., (x_h^{1*}, ..., x_h^{S*}), ..., b_h^*), ...)$ and Lagrange multipliers $(\lambda_h^{0*}, ..., (\lambda_h^{1*}/K, ..., \lambda_h^{S*}/K), ...) \in \Re_{++}^{\Sigma+1}$. We will repeatedly exploit this fact.

In the sequel, we will study equilibria making reference to the entire system of equations implicitly described in Definition 1, after getting rid of the redundant equations, i.e., to the so called *extended system of equations.* Let's define $F(..., (x_h, b_h, \lambda_h), ..., (p, q, \nu), ..., (u_h, e_h), ...)$ as

$$F(\xi, \nu, E) = \begin{bmatrix} ... \\ D_{x_h} V_h(x_h) - \lambda_h \Psi(p, \nu) \\ R(q)^T \lambda_h^T \\ -\Psi(p, \nu) z_h + R(q) b_h \\ .... \\ \sum_h z_h^{\backslash} \\ \sum_h b_h \end{bmatrix},$$

where $z_h^{\backslash}$ denotes agent h's vector of excess demand for all commodities but commodity C at each spot, while $\xi = (..., (x_h, b_h, \lambda_h), ..., (p, q))$ and $n = (H(G+I+\Sigma+1)+G^{\backslash}+I)$. $F(.)$ summarizes the first order conditions of each agent and, after applying appropriately Walras' law, the non-redundant market clearing conditions for assets and commodities as a function of the exogenous parameter ν, of the endogenous variables $\xi \in \Re^n$ and of endowments and utility functions. Evidently, F: $\Re^n \times \Re_{++}^{\Sigma-1} \times \mathcal{E} \longrightarrow \Re^n$. Let $F_E(\xi, \nu)$ be the map above for given $E \in \mathcal{E}$: Then, the set of equilibria of E is $F_E^{-1}(0)$.

3 Pareto improving sunspot equilibria with nominal assets

In this section, we consider economies with nominal assets and focus the analysis on the case of variable asset prices.

The basic idea of the proof is standard: The first step is to establish that, for a generic set of economies, k-invariant equilibria are regular and satisfy some additional restrictions (Lemma 2). Then, we show that these equilibria are Pareto dominated by some sunspot equilibria (Theorem 3).

The proof of the theorem requires us to be able to perturb independently the utility functions of two agents at each k-invariant equilibrium. In this class of economies, this is not necessarily possible, due to real indeterminacy. Therefore, we resort to fix $\nu \in \Re^{\Sigma-1}_{++}$: For such a given ν, we will establish the existence of an open and dense set of economies such that their k-invariant equilibria can be Pareto improved by changing appropriately ν.

In the proof of our main theorem, we need to restrict ourselves to regular equilibria where the matrix

$$\left[-Z^{\backslash 1} \cdots - Z^{\backslash K} \; -B \; -W^{\backslash} \right] =$$
$$\begin{bmatrix} \ldots & \ldots & \ldots \\ [\ldots, -\lambda_h^{\sigma} z_h^{\backslash\sigma}, \ldots]_{\sigma>0} & [-\lambda_h^0 b_h] & [\ldots, -\lambda_h^{\sigma} p^{\sigma} z_h^{\sigma}, \ldots]|_{\Sigma>\sigma>0} \\ \ldots & \ldots & \ldots \end{bmatrix},$$

evaluated at the k-invariant equilibrium, has maximal rank H.

Lemma 1. *Let $H < (S(C-2)+2)$ and $I < S$. Given $\nu \in \Re^{\Sigma-1}_{++}$, there is an open and dense subset of $\mathcal{E}$, $\mathcal{E}^g$, such that, for each $E \in \mathcal{E}^g$, at each k-invariant equilibrium,*

$$rank D_{\xi} F_E(\xi, \nu)|_{F_E(.)=0} = n,$$
$$rank \left[-Z^{\backslash 1} \cdots - Z^{\backslash K} \; -B \; -W^{\backslash} \right] = H.$$

Proof. The first result is basically established in [4], [14] and [16]. The second follows by a routine argument.

As is well known, the regularity part of the Lemma holds independently of S (and Σ).

The rank condition obviously implies that the equilibrium is Pareto inefficient. Similar rank conditions are common in the literature on constrained Pareto inefficiency. Our is slightly different (and stronger than the usual one) because we do not consider the $(C-1)$ columns of the excess demands for commodity $0c$, $c < C$. In terms of economic interpretation, bear in mind that the matrix is given by the gradients of the indirect utility functions with respect to $p^{\backslash\sigma}, \sigma > 0$, q and ν.

Given Lemma 2, we can restrict the analysis of the welfare effects of sunspot phenomena to regular sunspot equilibria, satisfying the additional rank condition above.

Using the standard approach, consider the system of equations

$$\Phi(\xi, \nu, E) = \begin{bmatrix} F(\xi, \nu, E) \\ V_1(x_1) - V_1^* \\ \ldots \\ V_H(x_H) - V_H^* \end{bmatrix}.$$

Suppose that, at the equilibrium ξ (associated with some ν),

$$rank D_{(\xi,\nu)} \Phi_E(\xi,\nu) = (n+H) .$$

Then, there exists a vector $(d\xi, d\nu)$ such that

$$D_{(\xi,\nu)} \Phi_E(\xi;\nu) \begin{bmatrix} d\xi \\ d\nu \end{bmatrix} = \begin{bmatrix} 0 \\ \eta \end{bmatrix},$$

for each $\eta >> 0$. Hence, by the implicit function theorem, we can find an appropriate perturbation of ν such that the associated equilibrium allocation Pareto dominates the original one. As mentioned above, our argument is an application of the approach originally introduced in [19] and developed and applied, in economies with incomplete markets, to issues related to constrained Pareto inefficiency (see, [8] and [7]) and to Pareto improving financial innovation (see, [5]).

Theorem 1. *Let $3 \leq H < (\Sigma - 1)$, $I < S$ and $K < (C-1)$. Then, for each $E \in \mathcal{E}^P$, an open, dense subset of $\mathcal{E}^g$, given $\nu = 1$, for each k-invariant equilibrium with allocation $x\prime$ there is a sunspot equilibrium with allocation $x\rfloor$ such that, for each h, $V_h(x_h^{"}) > V_h(x_h^{'})$.*

Evidently, $K < (C-1)$ and $H < (\Sigma - 1)$ imply that $H < (S(C-2)+2)$, so that Lemma 2 applies. We need at least three agents for technical reasons, related to the details of the proof of Lemma 7. It is an open issue if a similar result could be established for $H = 2$. As an immediate consequence of the theorem, we obtain

Corollary 1. *If $3 \leq H < (S-1)$ and $I < S$, then, for each $E \in \mathcal{E}^P$, an open, dense subset of $\mathcal{E}^g$, given $\nu = 1$, for each certainty equilibrium with allocation $x\prime$ there is a certainty equilibrium with allocation x" such that, for each h, $V_h(x_h^{"}) > V_h(x_h^{'})$.*

Remark 1. Both Theorem 3 and Corollary 4 take as a starting point a regular, k-invariant equilibrium. We can reinterpret the model as follows: Let h denote a type of agents and, therefore, assume that there is a continuum of agents of H different types. Then, our result holds true for economies with a continuum of agents, provided that the characteristics of agents are, a.e., not "too different" from the ones specified in the original H types (see, [5]).

Remark 2. Dealing with certainty equilibria, we do not face additional restrictions (but symmetry, negative semidefiniteness and continuity) on the perturbation of the Hessian matrix of the utility functions. Therefore, the result in Corollary 4 could also be established directly, by a straightforward application of the usual technique of proof (in fact, with $H \geq 2$). However, given the additional restrictions across extrinsic events that we must satisfy in sunspot economies, this technique cannot be applied in the more general framework of Theorem 3.

The proof of Theorem 3 is based on three steps: First, we observe that $rank D_{(\xi,\nu)}\Phi_E(\xi,\nu) = (n+H)$ if and only if

$$rank \begin{bmatrix} D_{(p^{\setminus},q)}\zeta & D_{\nu}\zeta \\ [-Z^{\setminus} \text{-}B] & [-W^{\setminus}] \end{bmatrix} = (G^{\setminus} + I + H),$$

where $\zeta(.)$ denotes the aggregate excess demand for all commodities but commodity C at each spot, while, now, $Z^{\setminus}$ is the $(H \times G)$ dimensional matrix with typical coefficient $(-\lambda_h^{\sigma} z_h^{\sigma c})$, each σ. Next, given $E = (u,e)$ with an equilibrium (p,q), (x,b) and with Lagrange multipliers λ, we pick an economy $\bar{E} = (u,\bar{e})$ such that the initial equilibrium is still an equilibrium of this modified economy and study $rank D_{(\xi,\nu)}\Phi_{\bar{E}}(\xi,\nu)$. We show that, generically in the space of the utility functions, $rank D_{(\xi,\nu)}\Phi_{\bar{E}}(\xi,\nu) = (n+H)$. In fact, to restore its full rank, it suffices to perturb the vector of marginal utilities of two agents. Then, a standard argument shows that (given the new collection of utility functions and modulo a perturbation of the original endowment e in the k-invariant direction $(e-\bar{e})$) $rank D_{(\xi,\nu)}\Phi_E(\xi,\nu) = (n+H)$.

We make precise our argument splitting it into three Lemma.

Let's start displaying $D_{(\xi,\nu)}\Phi_E(\xi,\nu)$,

$$\begin{bmatrix} \ddots & \cdots & \cdots & \cdots \\ \cdots & D_{(x_h,b_h,\lambda_h)}FOC_h & \cdots & D_{(p,q,\nu)}FOC_h \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \begin{bmatrix} I_G^{\setminus} & 0 & 0 \\ 0 & I_I & 0 \end{bmatrix} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & D_{x_h}V_h & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

In the matrix above,

$$D_{(x_h,b_h,\lambda_h)}FOC_h = \begin{bmatrix} D^2_{x_h}V_1 & 0 & -\Psi(p,\nu)^T \\ 0 & 0 & R(q)^T \\ -\Psi(p,v) & R(q) & 0 \end{bmatrix},$$

and

$$D_{(p,q,\nu)}FOC_h = \begin{bmatrix} -\Lambda_h^{\setminus}(\nu) & 0 & -\Xi(p,\lambda_h) \\ 0 & -\lambda_h^0 I_I & 0 \\ -\Psi(z_h^{\setminus},\nu) & \begin{bmatrix} -b_h \\ 0 \end{bmatrix} & -W_h^{\setminus} \end{bmatrix},$$

where

$$\Psi(z_h^{\setminus},\nu) = \begin{bmatrix} -z_h^{\setminus 0} & & & \\ & \ddots & & \\ & & -\nu^{\Sigma-1} z_h^{\setminus \Sigma-1} & \\ & & & -z_h^{\setminus \Sigma} \end{bmatrix}$$

is a $\left((\Sigma+1)\times G^{\setminus}\right)$ dimensional matrix,

$$W_h^{\setminus} = \begin{bmatrix} 0 & & 0 \\ p^1 z_h^1 & & \\ & \ddots & \\ & & p^{(\Sigma-1)} z_h^{(\Sigma-1)} \\ 0 & & 0 \end{bmatrix}$$

is a $((\Sigma+1)\times(\Sigma-1))$ dimensional matrix, $\Xi(p,\lambda_h)$ is a $(G+I)\times(\Sigma-1)$ dimensional matrix with typical column σ $(\sigma \neq 0, KS)$ given by $(0,...,\lambda_h^{\sigma} p^{\sigma}, 0, ...)^T$. Finally, $\Lambda_h^{\setminus}(\nu)$ is the $\left((G+I)\times(G^{\setminus}+I)\right)$ dimensional matrix with all σC rows identically zero. Restricted to the other rows, the matrix is diagonal with coefficients $\lambda_h^{\sigma}\nu^{\sigma}$.

Given that the block diagonal matrices $D_{(x_h,b_h,\lambda_h)}FOC_h$ are readily established to have full rank, $D_{(x,b,\lambda)}FOC^{-1}$ exists. By premultiplying $D_{(\xi,\nu)}\Phi$ by

$$\begin{bmatrix} -D_{(x,b,\lambda)}FOC^{-1} & 0 \\ 0 & I \end{bmatrix},$$

applying the obvious operations on the rows of the matrix so obtained and, finally, premultiplying by

$$\begin{bmatrix} -D_{(x,b,\lambda)}FOC & 0 \\ 0 & I \end{bmatrix},$$

we conclude that

$$rankD_{(\xi,\nu)}\Phi = rank \begin{bmatrix} D_{(x,b,\lambda)}FOC & D_{(p^{\setminus},q)}FOC & D_{\nu}FOC \\ 0 & D_{(p^{\setminus},q)}\zeta & D_{\nu}\zeta \\ 0 & \left[-Z^{\setminus} \; -B\right] & -W^{\setminus} \end{bmatrix}.$$

The next step is to compute the rank of the bottom right matrix for an appropriately selected economy $\bar{E}$. Notice that to keep in mind the entire *extended system of equations,* and, hence, the matrix $D_{(\xi,\nu)}\Phi(\xi,\nu,E)$, is essential to keep track of the exact effects that the operations on the columns of $\left[-Z^{\setminus} \; -B \; -W^{\setminus}\right]$ have on the columns of $D_{(p^{\setminus},q,\nu)}\zeta(.)$ via their effects on the columns of $D_{(p^{\setminus},q,\nu)}FOC_h(.)$.

Pick $\nu = 1$ and an associated k-invariant equilibrium (p^*, q^*) with allocation $(..., x_h^*, ...)$. We want to construct a new economy $\bar{E}$ with the same equilibrium prices and allocation, but with (k-invariant) individual endowments selected so that they allow us to drastically simplify the computations. Bear in mind that prices and allocations are, by assumption, k-invariant, so that we may sometime omit the index k, to emphasize this symmetry.

Let $\hat{s} = min\left\{s | s(C-2) \geq (H-2)\right\}$. The economy $\bar{E}$ is defined as follows: For each h, $\bar{u}_h = u_h$. Endowments are, instead, changed: For agent 1, set $\bar{e}_1^{sc} = x_1^{sc*}$ for all the commodities, but commodity $c = 1$ at each state $s, s > 0$, and commodity C at $s=0$. Set $\bar{e}_1^{0C}=\left(x_1^{0C*}+q^{1*}\right)$, $\bar{e}_1^{s1}=\left(x_1^{s1*}-1/p^{s1*}\right)$. Evidently, given $\bar{e}_1, (x_1^*, (1,0,...))$ is agent 1's optimal choice at prices (p^*, q^*). For agent $h, H>h>1$, define in the obvious way a one-to-one map

$$f:\{2,...,H-1\}\rightarrow\{12,\ ...,\ 1\,(C-1)\,,\ 22,\ ...,\ 2\,(C-1)\,,...,\ \hat{s}(C-1)\},$$

$f(h)=sc$. Then, set $\bar{e}_h^{sc}=x_h^{sc*}$ for each sc but $f(h)=sc$, and commodity C at the same spot. Set $\bar{e}_h^{sc}=(x_h^{sc*}-1)$, for $sc=f(h)$ and $\bar{e}_h^{sC}=\left(x_h^{sC*}+p^{f(h)*}\right)$. Evidently, given $\bar{e}_h$, $(x_h^*,0)$ is agent h's optimal choice at prices (p^*, q^*). His excess demand is nil for all commodities, but commodity $sc=f(h)$ and commodity C at the same spot and $z_h^{f(h)}=1$. For agent H, set $\bar{e}_H=\left(\sum_h x_h^*-\sum_{h<H}\bar{e}_h^*\right)$.

It is straightforward to check that, given $\nu=1$, (p^*, q^*) with allocation (x^*) is an equilibrium of the economy $\bar{E}$, supported by the portfolios $b_h=0$, for $h\neq 1, H$, and $b_1=-b_H=(1,0,...)$.

In words, agent 1 buys one unit of inside money and, at each spot, spends the revenue to buy commodity 1. At each $s\leq\hat{s}$, one unit of each commodity is traded with agent H by agent h such that $sc=f(h)$, who finances the purchase selling commodity C at the same spot. Also, each agent (but 1 and H) trades just once.

Observe that not necessarily $\bar{e}\in\Re_{++}^{(S+1)C}$. However, given that, at $\nu=1$ and in a neighborhood of (p^*, q^*), for each h and s, $p^{s*}\bar{e}_h^s>0$, this is irrelevant.

The result driving the entire proof is summarized in the next Lemma. The proof is in Appendix. The basic idea is the following: Given $\bar{E}$, the matrix $\left[-Z^{\setminus}\ -B\ -W^{\setminus}\right]$ has a very simple structure. In particular, using columns operations, it can be transformed into a matrix $\left[-Z^{\setminus*}\ 0\ 0\right]$ where $Z^{\setminus 0*}=0$, while each block $Z^{\setminus k*}$ contains only one nonzero term, in the column corresponding to commodity (ksk). Of course, these columns operations also affect the matrix $D_{(p^{\setminus},q,\nu)}\zeta(.)$, transforming it into a matrix $D_{(p^{\setminus},q,\nu)}\zeta^*(.)$. Let p° be the vector of prices of the commodities p^{ksk}, each k. The key step of the proof is to show that, given u_1 and modulo a perturbation of u_H and u_2, the $(G^{\setminus}+I)$ columns of $D_{(p^{\setminus},q,\nu)}\zeta^*(.)$ referred to $(p^{\setminus\circ}, q, \nu)$, where the vector $p^{\setminus\circ}$ does not includes commodity ksk, each k, are linearly independent. When this is true, relabelling columns, we obtain

$$rank\begin{bmatrix} D_{(p^{\setminus},q)}\zeta & D_{\nu}\zeta \\ \left[-Z^{\setminus}\ -B\right] & -W^{\setminus}\end{bmatrix}=rank\begin{bmatrix} D_{(p^{\setminus\circ},q,v)}\zeta^* & D_{p^{\circ}}\zeta^* \\ 0 & -Z^{\setminus*}\end{bmatrix}=\left(G^{\setminus}+I+H\right).$$

This sketch of the proof should help to understand why we need a restriction on the number of extrinsic events : We need to replace column $D_{p^{ksk}}\zeta^*$ with column $D_{\nu^{ks}}\zeta^*$, each $ks\neq KS$. This works as long as $K<(C-1)$. The economy $\bar{E}^{\delta}$ constructed in the Lemma depends upon the particular equilibrium we start with. However, for given ν, $F_E^{-1}(0)$ is a finite set (generically). Hence, iterating the procedure, we can show that the same result holds for each equilibrium, associate with the given ν, of an open and dense set of economies.

Lemma 2. *Let* $3 \leq H < (\Sigma - 1)$, $I < \Sigma$ *and* $K < (C-1)$. *Given* ν, $\bar{E}$ *and any open set* $V(\bar{E})$, *there is* $\bar{E}^{\delta} = (u^{\delta}, \bar{e}) \in V(\bar{E})$ *with the same equilibrium* (p,q) *and* (x,b) *and such that* $rankD_{(\xi,\nu)}\Phi_{\bar{E}^{\delta}} = (n+H)$.

Now, let's go back to the actual economy E. Here, we exploit an argument which has been introduced by [1] and subsequently used, in dealing with sunspot economies, by [4] and [10].

Lemma 3. *Let* $3 \leq H < (\Sigma - 1), I < \Sigma$ *and* $K < (C-1)$. *Given* ν, E *and any open set* $V(E)$, *there is an open, dense subset* $V\prime(E) \subset V(E)$, *such that, for each* $E^{'} \in V\prime(E)$, $rankD_{(\xi,\nu)}\Phi_{E'}|_{F(.)=0} = (n+H)$.

Proof. Start with the economy E and construct $\bar{E}^{\delta} = (u^{\delta}, \bar{e})$. By Lemma 7, $rankD_{(\xi,\nu)}\Phi_{\bar{E}^{\delta}} = (n+H)$. In fact, in Appendix, we establish the result dropping the redundant columns of $D_{\nu}\Phi_{\bar{E}^{\delta}}$, so that we obtain a square matrix $D_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^{\delta}}$. Given that $rankD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^{\delta}} = (n+H)$, $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^{\delta}} \neq 0$. Let $e(t) = (t\bar{e} + (1-t)e)$ and $\bar{E}^{\delta}(t) = (u^{\delta}, e(t))$. Evidently, $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^{\delta}(t)}$ is a polynomial in the variable t. Given that $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^{\delta}(1)} \neq 0$, the polynomial is non trivial and, therefore, $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^{\delta}(t)} = 0$ has a finite number of solutions, $\{t_1, ..., t_T\}$. Hence, if $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^{\delta}(0)} = 0$, it is sufficient to perturb the initial endowment in the (k-invariant) direction $(\bar{e}-e)$ to obtain an (arbitrarily close) economy $E\prime = (u^{\delta}, e\prime)$ with $detD_{(\xi,\nu)}\Phi^{\setminus}|_{E'} \neq 0$. Hence, $V\prime(E)$ is dense. Given that the initial equilibrium is regular in the economy E, openness of $V\prime(E)$ follows immediately, for $V(E)$ sufficiently small.

We are finally ready to establish Theorem 3.

Proof. Density of $\mathcal{E}^P$ follows directly from Lemma 2, 7 and 8. Given the boundary conditions, a standard argument establishes that $\mathcal{E}^P$ is open as well.

4 Pareto improving sunspot equilibria with real assets

As is well known, while sunspot equilibria may exist in economies with real assets, they do not exist for arbitrarily given payoffs structures. We will extend the previous result showing that, in economies with incomplete markets (at the k-invariant equilibrium) and such that sunspot equilibria exist, there are, generically, asset payoffs structures such that the original equilibrium is Pareto dominated. To guarantee the existence of sunspot equilibria, we will follow the approach originally proposed in [13] and developed in [9].

Our construction rests heavily on the previous results. Pick an economy E, nominal asset structure Y and ν' such that the associated equilibrium (more properly, one of the equilibria) $(p\prime, q\prime)$ with allocation $(x\prime, b\prime)$ is Pareto dominated by a sunspot equilibrium associated with ν", $(p", q")$, with allocation

$(x", b")$. We will show that, typically, there is a k-invariant structure of real assets ρ such that $(p\prime, q\prime)$, $(x\prime, b\prime)$ and $(p", q")$, $(x", b")$ are both equilibria of the new economy.

Notice that, by Corollary 4, if $H < (S-1)$ and assets are nominal, certainty equilibria are typically Pareto dominated by some other certainty equilibrium, so that the possibility of Pareto improvements within the set of equilibria is not necessarily related to sunspots phenomena. This is in general not true with real assets. It is easy to check that our construction works also for economies with a unique certainty equilibrium.

The proof of the main result exploits an additional property of sunspot equilibria. Define the $((K+1) \times C)$ dimensional matrix

$$\Xi(s) = \begin{bmatrix} \nu^{s'} p^{s'} \\ \nu^{1s"} p^{1s"} \\ \ldots \\ \nu^{Ks"} p^{Ks"} \end{bmatrix}$$

where the first row is given by state s certainty equilibrium prices at ν', while the last K rows are given by equilibrium prices at the states (ks) at ν''.

Lemma 4. *Let* $3 \leq H < (\Sigma - 1)$, $I < S$ *and* $K < (C-2)$. *Given the nominal asset structure* Y, *let* $(p\prime, q\prime)$ *be the k-invariant equilibrium associated with* ν' *and* $(p", q")$ *be a sunspot equilibrium associated with* $\nu"$, *satisfying* $v^{k_a s"} \neq v^{k_b s"}$ *for each s and each pair* k_a, k_b. *Then, for an open and dense set* $\mathcal{E}^* \subset \mathcal{E}$, $\Xi(s)$ *has maximal rank* $(K+1)$, *for each s.*

Proof. By Lemma 2, there is no loss of generality in assuming that both k-invariant and sunspot equilibria are regular and sufficiently close (in the economy with nominal assets). Hence, it suffices to show the density part of the thesis, i.e., that for each open ball $V(E)$, there is $E^* \in V(E)$ such that $rank\Xi(s) = (K+1)$, for each s, at the equilibrium (p^*, q^*). By a standard argument, modulo a perturbation of the utility function in the certainty economy, there is an h, say $h = 1$, such that, at each σ, $p^{\sigma'} z_1^{\sigma'} \neq 0$. By regularity of the sunspot equilibrium and continuity, we can assume that $p^{\sigma"} z_1^{\sigma"} \neq 0$, at each σ. Given that $v^{k_a s"} \neq v^{k_b s"}$, each s and each pair k_a, k_b, this implies $x_1^{k_a s"} \neq x_1^{k_b s"}$, each s and each pair k_a, k_b, and $\mathrm{x}_1^{s'} \neq x_1^{ks"}$, each k. Hence, there exists a collection of K open balls $B_1(x_1^{k"})$, each one centered on $x_1^{k"} = (x_1^{0"}, x_1^{k1"}, ..., x_1^{kS"})$, $k = 1, ..., K$, and a collection of closed balls $B_1^{\varepsilon}(x_1^{k"}) \subset \Re_{++}^{(S+1)C}$ such that $B_1(x_1^{k"}) \subset B_1^{\varepsilon}(x_1^{k"}) \subset \Re_{++}^{(S+1)C}$, each k, and $B_1^{\varepsilon}(x_1^{k_a"}) \cap B_1^{\varepsilon}(x_1^{k_b"}) = \emptyset$, each k_a, k_b and $x_1^{'} \notin B_1^{\varepsilon}(x_1^{k"})$, each k. Also, let $\Theta^k(x_1, B_1^{\varepsilon}(x_1^{k"}))$, $k = 1, ..., K$, be a collection of bump functions taking the value 1 at $x_1 \in B_1(x_1^{k"})$, the value 0 at $x_1 \notin B_1^{\varepsilon}(x_1^{k"})$ (see, [12]). Assume that $rank\Xi(s) < (K+1)$, for some s. Consider an arbitrarily small perturbation of its last K rows (keeping $p^{ks1} = 1$, each k), so that the new matrix $\Xi'(s)$ has full rank. Repeat the operation for each $s > 0$. Let p^*

be the price vector so obtained and let $\Delta^*(p^*, q^*, \nu") = \sum_{h>1} z_h^\sigma(p^*, q^*, \nu")$ and $z_1^* = z_1(p^*, q^*, \nu")$. Given that $\sum_h z_h^\sigma(p", q", \nu") = 0$, by continuity, $|| -\Delta(p^*, q^*, \nu") - z_1(p^*, q^*, \nu")||$ can be made arbitrarily close to 0 by choosing a price vector p^* sufficiently close to $p"$ and still preserving the full rank of $\Xi'(s)$, each s. Consider now the economy E^* with $e_h^* = e_h$, each h, $u_h^* = u_h$, $h > 1$, and $u_1^*(x_1) = u_1(x_1) + \sum_k \Theta^k(x_1, B_1^\varepsilon(x_1^{k"}))[D_{x_1}u_1(x_1)|_{z_1^*} - D_{x_1}u_1(x_1)|_{-\Delta^*}]x_1$. I claim that, $(-\Delta^*, b_1^* = -\sum_{h>1} b_h(p^*, q^*, \nu"))$ is agent 1's optimal choice given (p^*, q^*), $\nu"$ and utility function u_1^*. Evidently, $(-\Delta^*, b_1^*)$ satisfies agent 1's sequence of budget constraints. Moreover, given that, with utility u_1, at z_1^*, $\sum_k \pi(k) D_{x_1}u_1(x_1)|_{z_1^*} = \lambda_1^{*T}\Psi(p^*, \nu")$, for some vector λ^* such that $\lambda_1^{*T}R(q^*) = 0$, $\sum_k \pi(k) D_{x_1}u_1^*(x_1^k)|_{-\Delta^*} = \sum_k \pi(k) D_{x_1}u_1(x_1^k)|_{-\Delta^*} + \sum_k \pi(k)[D_{x_1}u_1(x_1)|_{z_1^*} - D_{x_1}u_1(x_1)|_{-\Delta^*}] = \sum_k \pi(k)[D_{x_1}u_1(x_1)|_{z_1^*} = \lambda_1^{*T}\Psi(p^*, \nu")$. Hence, $(-\Delta^*, b_1^*)$ solves the FOC of agent 1's optimization problem at (p^*, q^*) and $\nu"$. By construction, the allocation $(z_1 = -\Delta^*, z_2(p^*, q^*, \nu"), ..., z_H(p^*, q^*, \nu"))$ satisfies the market clearing conditions. Given that Y is a full rank matrix, this means that the market clearing conditions for assets are satisfied as well. Hence, (p^*, q^*) is a sunspot equilibrium of the economy E^*, given $\nu"$. Moreover, given that agent 1's utility function is not changed at x_1', the k-invariant equilibrium is not affected. As noticed above, for a sufficiently small perturbation of the equilibrium prices $p"$, E^* can be made arbitrarily close to E, hence $E^* \in V(E)$.

Finally,

Theorem 2. *Let $3 \leq H < (\Sigma - 1)$, $I < S$ and $K < (C - 2)$. Then, there exists an open, dense set of economies $\mathcal{E}^p \subset \mathcal{E}$ such that, for each $E \in \mathcal{E}^P$, there is an asset structure ρ such that there are sunspot equilibria Pareto superior to a k-invariant equilibrium.*

Proof. Given Lemma 9, it suffices to establish the density part. Given E, any open set $V(E)$ and any asset structure Y, with $I < S$, pick E^* such that the regular k-invariant equilibrium associated with ν is Pareto dominated by a sunspot equilibrium associated with $\nu"$. Given the previous Lemma, we may assume that, at the sunspot equilibrium $p"$, $rank\Xi(s) = (K+1)$, each s. This implies that, for each s and each i, there is a solution $\rho^{i*}(s)$ to the system of equations $\Xi(s)\rho^i(s) = [y^{is}, y^{is}, ..., y^{is}]$. Then, let ρ^* define the structure of real asset of the economy. Evidently, $(p\prime, q\prime)$ is a k-invariant equilibrium, while $(p", q")$ is a sunspot equilibrium of the new economy.

Remark 3. In the economy with real assets, neither k-invariant nor sunspot equilibria are necessarily regular (because $D_{(p,q)}\zeta(.)$ in the economy with real assets is obviously different from the same derivative in the economy with nominal assets). Given [9], our result could be strenghtened by considering equilibria which are regular in the economy with real assets, too. This would allow to strenghten Theorem 10, establishing openness of the set of asset structures allowing for Pareto improvements. The argument is straightforward and, therefore, omitted.

Appendix: Proof of Lemma 7

Throughout the proof, we will keep fixed $\nu = 1$ and an associated equilibrium (p, q) and perturb the economy $\bar{E}$ so that $(p, q) \in F_{\bar{E}'}^{-1}(0)$. The technique of proof requires us to be able to perturb (in k-invariant directions) the gradients of two agents. It is well known that this can be obtained via perturbations of the utility functions (see, [13], [14] and [16]). The following result is reported here for completeness.

Claim. Let $I < S$. Fix ν^*, $\bar{E}$, any open set $V(\bar{E})$ and a k-invariant equilibrium (p^*, q^*), (x^*, b^*) with associate Lagrange multipliers λ^*. Then, for each $\delta \in \Re^{\Sigma+1}$, δ small enough and such that $R(q)\delta^T = 0$, there is $\bar{E}^\delta \in V(\bar{E})$ such that (p^*, q^*), (x^*, b^*) is an equilibrium of $\bar{E}^\delta$ with associated Lagrange multipliers $\lambda_h^\delta = (\lambda_h^* + \delta)$.

The proof of Lemma 7 is based on two steps:

1. We exploit the k-invariance of the allocation to get rid of all but H linearly independent columns of the submatrix $[Z^\setminus \; B \; W^\setminus]$. Given u_1, this may require a perturbation of agent H's utility function.
2. We show that, given u_1, u_H and modulo an arbitrarily small perturbation of u_2, the $(G^\setminus + I)$ dimensional, square matrix $D_{(p^{\setminus\circ}, q, \nu)}\zeta$, defined in the text above, has full rank.

The Lemma follows immediately.

Step 1

Bear in mind that $K < (C-1)$ and $H < (\Sigma - 1)$.

Without loss of generality, but for notational simplicity, fix $\nu^{ks} = 1$ for each $s > \hat{s}$ and drop the corresponding columns of $W^\setminus$ and $D_\nu\zeta$.

Remember that $\hat{s} = \; min\{s | s(C-2) \geq (H-2)\})$ and, without any loss of generality, assume $\hat{s}(C-2) = (H-2)$.

Also, for notational convenience, let $\ell_h = K(\lambda_h^{11}, ..., \lambda_h^{KS})/\lambda_h^0$. Hence, if $\ell_h^c = (\ell_h^1, ..., \ell_h^S)$ is the vector of period 1, normalized (by λ_h^0) Lagrange multipliers at the certainty equilibrium, $\ell_h = (..., \ell_h^c, ...)$ is the normalized (by λ_h^0/K) period 1 vector of Lagrange multipliers at the associate k-invariant equilibrium.

In the economy $\bar{E}$ (and after normalizing row h by λ_h^0/K, each h), $Z^\setminus$ is given by $\left[KZ^{\setminus 0} \; Z^{\setminus 1} \; ... \; Z^{\setminus K} \right]$, with $Z^{\setminus 0} = [0]$, while $-Z^{\setminus k} =$

$$\begin{bmatrix} -\ell_1^1/p^{11} & \cdots & 0 & -\ell_1^2/p^{21} & \cdots & \cdots & \cdots & -\ell_1^S/p^{S1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -\ell_{C-1}^1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -\ell_{H-1}^{\hat{s}} & \cdots & 0 \\ \ell_H^1/p^{11} & \cdots & \ell_H^1 & \ell_H^2/p^{21} & \cdots & \ell_H^{\hat{s}} & \cdots & \ell_H^S/p^{S1} \end{bmatrix},$$

and

$$\begin{bmatrix} -(KB) & -W \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -K & 0 \\ \cdots & \cdots \\ K & 0 \end{bmatrix} & \begin{bmatrix} -\ell_1^1 & & -\ell_1^{\hat{s}} \\ & \ddots & \\ \ell_H^1 & & \ell_H^{\hat{s}} \end{bmatrix} \end{bmatrix}.$$

Evidently, there is no loss of generality in assuming that $Z^{\setminus}$ has maximal rank H: Relabelling spots and agents, it suffices to pick agent 1 and H so that $\ell_1^s/\ell_1^{s'} \neq \ell_H^s/\ell_H^{s'}$, for some s and $s\prime$. With reference to the previous claim, set $\delta_H = \left[\delta_H^0, \delta_H^1, 0, \delta_H^3, ..., \delta_H^S\right]$, $\delta_H^1 \neq 0$, and δ_H^s, $s \neq 1, 2$, chosen so that $R(q)\delta_H = 0$, which can always be done because $\left[\, y^{k1} \cdots y^{kS} \,\right]$ is in general position, for each k.

Consider the following sequence of column operations (we identify columns in the obvious way), focusing, for the moment, on their effects on the columns of $\left[\, Z^{\setminus}\ (KB)\ W^{\setminus} \,\right]$:

a. For each s, $\hat{s} \geq s > 0$, subtract from column (ν^{ks}) column (p^{ks1}) multiplied by p^{s1} (i.e., get rid of the nonzero coefficients of the matrix $W^{\setminus}$).

b. Subtract column (p^{ksc}), $c = k$, from column $(p^{k'sk})$ to eliminate all the (collinear) columns $(p^{k'sk})$, $k\prime \neq k$.

c. Use the linearly independent columns (p^{111}) and (p^{121}) to eliminate columns (p^{1s1}), $s > 2$, and the nonzero columns of the matrix B.

Step 2:

Rearrange the columns of the (transformed) matrix to obtain

$$\begin{bmatrix} D_{(p^{\setminus\circ}, q, \nu)}\zeta^* & D_{p^\circ}\zeta^* \\ 0 & Z^* \end{bmatrix},$$

where Z^* has full rank H. The square matrix $D_{(p^{\setminus\circ}, q, \nu)}\zeta^*$ is given by the columns (modified by the operations under a, b and c above) relative to (p^{0c}), $c < C, (p^{ksc})$, each $ksc \neq ksk$ and $c < C$, and (q^i). The other H columns relative to (p^{ksk}), $s \leq \hat{s}$, have been replaced by the columns (ν^{ks}).

We need to show that, generically, rank $D_{(p^{\setminus\circ}, q, \nu)}\zeta^* = \left(G^{\setminus} + I\right)$.

Let $[D_{(x_h, b_h, \lambda_h)}FOC_h]^{\setminus -1}$ be the matrix obtained from $[D_{(x_h, b_h, \lambda_h)}FOC_h]^{-1}$ deleting the rows associated with commodity σC, each σ, and the last $(\Sigma + 1)$ rows. By the implicit function theorem,

$$D_{(p^{\setminus},q,\nu)}\ z_h^{\setminus} = -[D_{(x_h,b_h,\lambda_h)}FOC_h]^{\setminus -1}D_{(p^{\setminus},q,\nu)}FOC_h.$$

Let $D_{(p^{\setminus\circ},q,\nu)}FOC_h^*$ be the matrix obtained from $D_{(p^{\setminus},q,\nu)}FOC_h$ with the operations performed in step 1 and the substitution of columns just described. Then, $D_{(p^{\setminus\circ},q,\nu)}\zeta^* = -\sum_h[D_{(x_h,b_h,\lambda_h)}FOC_h]^{\setminus -1}D_{(p^{\setminus\circ},q,\nu)}FOC_h^*$.

Claim. Assume that $[D_{(x_h,b_h,\lambda_h)}FOC_h]^{\setminus -1}D_{(p^{\setminus\circ},q,\nu)}FOC_h^* = \lambda_h^0 M_h$, where M_h is a square $(G^{\setminus}+I)$ dimensional matrix of full rank, which does not depend upon λ_h^0. Then, given any open set $V(\bar{E})$, there is $\bar{E}^1 \in V(\bar{E})$ such that $(p,q) \in F^{-1}(0)$ and $rankD_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}^1} = (G^{\setminus}+I)$.

Proof. Given the previous claim, set $\delta_h = -\lambda_h\eta$. Then, agent h's vector of Lagrange multipliers becomes $\lambda_h(1-\eta)$: Hence, the normalized vector does not change, while $\lambda_h^{0\delta} = \lambda_h^0(1-\eta)$.

Then, $D_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}^\delta} = D_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}} - (\eta\lambda_h^0)M_h$ and, therefore, $(\eta\lambda_h^0)$ is an eigenvalue of the matrix $M_h^{-1}D_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}}$. The set of eigenvalues of any matrix is finite, hence, for almost all η, $rankD_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}^\delta} = (G^{\setminus}+I)$.

Therefore, to conclude, we just need to show $rankM_h = \left(G^{\setminus}+I\right)$, for some h. Pick $h=2$ (notice that, as it will become clear in the sequel, it needs to be $h \neq 1, H$. That's why we need $H > 2$) and let

$$[D_{(x_2,b_2,\lambda_2)}FOC_2]^{-1} = \begin{bmatrix} \varDelta_2^{1*} & \varDelta_2^{2T} \\ \varDelta_2^2 & \varDelta_2^3 \end{bmatrix},$$

where $\varDelta_2^{1*}$ is well known to be a $(G+I)$ dimensional square matrix of rank $(G^{\setminus}+I)$. Also, let $\varDelta_2^1$ be the $\left((G^{\setminus}+I)\times(G+I)\right)$ dimensional matrix obtained deleting rows σC, each σ. It is well-known that $rank\varDelta_2^1 = (G^{\setminus}+I)$ (see, for instance, Balasko and Cass (1991)). Moreover,

$$\varDelta_2^1\left[-\varPsi(p,\nu)\ R(q)\right]^T = 0$$

so that $\left[-\varPsi(p,\nu)\ R(q)\right]^T$ spans the null space of $\varDelta_2^1$. To conclude the proof, first we observe that the last $(\varSigma+1)$ rows of $D_{(p^{\setminus\circ},q,\nu)}FOC_2^*$ are identically zero and that $D_{(p^{\setminus\circ},q,\nu)}FOC_2^*$ is homogeneous of degree 1 in λ_2^0, so that we can write

$$D_{(p^{\setminus\circ},q,\nu)}FOC_2^* = \begin{bmatrix} \lambda_2^0 D_2 \\ 0 \end{bmatrix}$$

and, therefore, $[D_{(x_1,b_1,\lambda_1)}FOC_2]^{\setminus -1}D_{(p^{\setminus\circ},q,\nu)}FOC_2^* = \lambda_2^0\varDelta_2^1 D_2$. Finally, we show that:

a. D_2 has full column rank;

b. $spanD_2 \cap span\left[-\varPsi(p,\nu)\ R(q)\right]^T = \{0\}$.

It follows that $B_2^1\varDelta_2 = M_2$ has full rank $\left(G^{\setminus}+I\right)$, as required.

Bear in mind that, at a k-invariant equilibrium, p^{ks} and ℓ_2^{ks} are k-invariant, so that we can omit the index k, to stress the symmetry.

Consider the changes induced in the matrix $D_{(p^{\setminus},q,\nu)}FOC_2$ by the sequence of operations on the columns and by their final rearranging. The first $(G^{\setminus}+I)$ columns of the modified matrix (using the normalization by K/λ_2^0) are given by the matrix $(\lambda_2^0/K)D_2^*$, whose last $(\Sigma+1)$ rows are identically zero (in step 1, we got rid of all the non-zero terms of these rows but $-z_2^{11}$ and $-z_2^{21}$ and the two corresponding columns have been replaced by the ones associated with ν^{11} and ν^{12}). Indeed, $D_2^* =$

$$\begin{bmatrix} -KI^{\setminus} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & -A^{11} & \cdots & \cdots & C_S^{11} & \cdots & \cdots & C_I^{11} \\ \cdots & \cdots & -A^{12} & \cdots & C_S^{12} & \cdots & \cdots & C_I^{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & -A^{1S} & \cdots & T^{1S} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -A^{KS} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -KI_I \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

where:

$$A^{ks} = \begin{bmatrix} \ell_2^s & & 0 & & \\ & \ddots & & & \\ & & \ell_2^s p^{ksk} & & \\ & & & \ddots & \\ & & & & \ell_2^s \\ & & \ell_2^s & & 0 \end{bmatrix},$$

i.e., the vector $\begin{bmatrix} 0 & \ell_2^s p^{s2} & \cdots & \ell_2^s \end{bmatrix}^T$ replaces the column referred to commodity $c = k$ (remember that we subtracted from column ν^{ks} column (p^{1s1}) multiplied by p^{s1} in step 1a and, then, replaced column (p^{ksk}) with the (modified) column ν^{ks}).

The matrices T^{ks}, $s \leq \hat{s}$, are given by all zero coefficients, but the coefficient corresponding to commodity $c = \ k$, which is equal to ℓ_2^s (remember step 1b above).

The matrices C_s^{1t} and C_I^{1t}, $t = 1,2$, $s > 2$, reflect the operations of step 1c. Then, for $s > 2$, $t = 1,2$, C_s^{1t} has all the coefficients equal to zero, but the ones on the first row and column, given by $\ell_2^t c_s^{1t}$, where

$$\begin{bmatrix} c_s^{11} \\ c_s^{12} \end{bmatrix} = \begin{bmatrix} -\ell_1^1/p^{11} & -\ell_1^2/p^{21} \\ \ell_H^1/p^{11} & \ell_H^2/p^{21} \end{bmatrix}^{-1} \begin{bmatrix} -\ell_1^s/p^{s1} \\ \ell_H^s/p^{s1} \end{bmatrix}.$$

Similarly, C_I^{1t}, $t = 1,2$, has all the coefficients equal to zero, but the ones on the first row and column, given by $\ell_2^t c_I^{1t}$,where

$$\begin{bmatrix} c_I^{11} \\ c_I^{12} \end{bmatrix} = \begin{bmatrix} -\ell_1^1/p^{11} & -\ell_1^2/p^{21} \\ \ell_H^1/p^{11} & \ell_H^2/p^{21} \end{bmatrix}^{-1} \begin{bmatrix} -K \\ K \end{bmatrix}.$$

Bear in mind that these matrices just appear on the blocks of rows of states 11 and 12, given the particular role played by the value of the excess demands of agents 1 and H in these two states.

Let D_2 be the $\left((G+I)\times(G^{\setminus}+I)\right)$ dimensional matrix given by the first $(G+I)$ rows of D_2^*. Given that $\nu = 1$, for the purpose of this computation, we can drop it from the notation.

Claim. D_2 has full column rank.

Proof. Suppose that $D_2 a^T = 0$, and let $a = (a^0, a^1, ..., a^K, a^{\Im}) \in \Re^{G^{\setminus}+I}$, with $a^{\Im} = (a^{\Im 1}, ..., a^{\Im I})$. Evidently, $a^0 = 0$, $a^{\Im} = 0$ and $a^{ks} = 0$, for each $s > \hat{s}$. For a^{ks}, $k >$ 1, given the structure of A^{ks}, $a^{ksk} = 0$, because for each ksk there is a row (the one corresponding to commodity ksC) with one and only one non zero coefficient. This immediately implies that $a^{ksc} = 0$, each c. For $k = 1$, the same argument applies to blocks ks, $s > 2$. Taking into account these facts and given that A^{11} has full rank, $a^{11} = a^{12} = 0$.

Claim. Let $I \ < S$, and $3 \leq H$. Then, for each open set $V(\bar{E})$, there is an open and dense subset $V\prime(\bar{E}) \subset V(\bar{E})$ such that, for each $\bar{E}' \in V'(\bar{E})$, $span D_2 \cap span\left[-\Psi(p)\ R(q)\right]^T = \{0\}$.

Proof. Openness follows by regularity of the equilibrium (if $V(\bar{E})$ is sufficiently small). Hence we just need to show the density part.

If $b \in span\left[-\Psi(p)\ R(q)\right]^T$, then
$b = \left[-b^0p^0, (-b^{11}p^{11}, ..., -b^{1S}p^{1S}), ..., (-b^{K1}p^{11}, ..., -b^{KS}p^{1S}), b^{\Im}\right]$,
with $b^{\Im} = R(q)^T\left[b^0, ..., b^K\right]^T$.

As above, let $a = \left[a^0, a^1, a^K, a^{\Im}\right] \in \Re^{G^{\setminus}+I}$ and, by contradiction, assume that there is $b \in \Re^{G+I} \setminus \{0\}$ such that $b \in span\left[-\Psi(p)^T\ R(q)^T\right]$ and $b = D_2 a$, for some a. The structure of b imposes several restrictions on the coefficients of the vector a. We are going to show that, generically in u_2, they are satisfied if and only if $a = 0$. Hence, $b = 0$ and the contradiction establishes the claim.

Observe that:

a. Given that $-b^0p^0 = -\left[a^{01}K, ..., a^{0C-1}K, 0\right]$, $a^0 = b^0 = 0$.

b. For states 11 and 12, given that, for $t = 1, 2$, $-b^{1t}p^{1t} =$ $[\ell_2^t\left(a^{\Im 1}c_I^{1t} + \sum_{s>2} a^{1s1}c_s^{1t} + \sum_{k>1} a^{kt1}\right), ..., -\ell_2^t(a^{1t1}p^{tc} + a^{1tc}), ..., -\ell_2^t a^{1t1}]$, it must be $b^{1t} = \ell_2^t a^{1t1}$. Hence, $a^{1tc} = 0$, $C > c > 1$, and $\left(a^{\Im 1}c_I^{1t} + \sum_{s>2} a^{1s1}c_s^{1t} + \sum_{k>1} a^{kt1}\right) = -a^{1t1}p^{t1}$.

c. For b^{1s}, $\hat{s} \geq\ s > 2$, $-b^{1s}p^{1s} = [\ell_2^s \sum_{k>1} a^{ks1}, -(\ell_2^s a^{1s1}p^{s2} + \ell_2^s a^{1s2}), ..., -\ell_2^s a^{1s1}]$. Hence, it must be $b^{1s} = \ell_2^s a^{1s1}$ and, therefore, $a^{1sc} = 0$, $c > 1$, and $\sum_{k>1} a^{ks1} = \text{-}a^{1s1}p^{s1}$.

d. For $k > 1$, and $s \leq \hat{s}$, $-b^{ks}p^{ks} = -\left[\ell_2^s a^{ks1}, .., \ell_2^s a^{ksk} p^{sc}, ..., \ell_2^s a^{ksk}\right] + \left[0, -\ell_2^s a^{ks2}, ..., \ell_2^s \sum_{k' \neq k} a^{k'sk'}, ..., -\ell_2^s a^{ksC-1}, 0\right]$. Hence, $b^{ks} = \ell_2^s a^{ksk}$, and, therefore, $a^{ksc} = 0$, for $c \neq 1, k$, $a^{ks1} = a^{ksk}p^{s1}$ and $\sum_{k' \neq k} a^{k'sk'} = 0$.

e. For s $\geq \hat{s}$, given that $-b^{ks}p^{ks} = -\ell_1^s(a^{ks}, 0)$, it must be $a^{ks} = b^{ks} = 0$, each k.

f. Given that $-K\left[a^{\Im}\right]^T = b^{\Im T} = R(q)^T\left[b^0, ..., b^K\right]^T$, and taking into consideration what we have just established, it must be

$-K\left[a^{\Im}\right]^T = R(q)^T\left[0 \; \ell_2^1 a^{11} \cdots \ell_2^S a^{KS}\right]^T$.

Bear in mind that, given that (by d) $a^{ksc} = 0$, for all c, but $c = 1, k$ (each s and $k > 1$ and each $1s, s > 2$), and that $a^{ks1} = a^{ksk}p^{s1}$, we can denote a^{ksk} as a^{ks}, for each $s, k > 1$, and for $k = 1, s > 2$.

To summarize, we must have:

i. $a^0 = a^{ks} = 0$, for $s > \hat{s}$ and each k (because of a and e);

ii. $\left(a^{\Im 1}c_I^{1t}/p^{t1} + \sum_{s>2} a^{1s}c_1^{1t}/p^{t1} + \sum_{k>1} a^{kt}\right) = -a^{1t}$, $t = 1, 2$ (because of b and remembering that $a^{ks1} = a^{ksk}p^{s1} = a^{ks}$, for $s > 2$ and each $ks, k > 1$);

iii. $\sum_{k>1} a^{ks} = -a^{1s}$, $s > 2$ (because of c);

iv. $\sum_{k' \neq k} a^{k's} = 0$, $k > 1$ and each s (because of d);

v. $-K\left[a^{\Im}\right]^T = R(q)^T\left[0 \; \ell_2^1 a^{11} \cdots \ell_2^S a^{KS}\right]^T$ (this is f).

Ignore the subset of states such that it must be $a^{ks} = 0$. Translating these restrictions on the vector a into matrix form, it must be

$$\begin{bmatrix} I_{\hat{s}} + \Upsilon & I_{\hat{s}} & \cdots & I_{\hat{s}} & C \\ I_{\hat{s}} & 0 & \cdots & I_{\hat{s}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{\hat{s}} & I_{\hat{s}} & \cdots & 0 & 0 \\ Y(1)^{\setminus T} D\left(\ell_2^{\setminus}\right) & Y(2)^{\setminus T} D\left(\ell_2^{\setminus}\right) & \cdots & Y(K)^{\setminus T} D\left(\ell_2^{\setminus}\right) & K I_I \end{bmatrix} a^T$$

$= V a^T = 0$,

where, now, $a = (..., \left(a^{k1}, ..., a^{k\hat{s}}\right), ..., a^{\Im}) \in \Re^{\hat{s}K+I}$, C is the $(\hat{s} \times I)$ matrix

$$\begin{bmatrix} c_I^{11}/p_1^{11} & \cdots & 0 \\ c_I^{12}/p^{21} & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix},$$

while Υ is the $(\hat{s} \times \hat{s})$ matrix

$$\begin{bmatrix} 0 & 0 & c_3^{11}/p^{11} & \cdots & c_{\hat{s}}^{11}/p^{11} \\ 0 & 0 & c_3^{12}/p^{21} & \cdots & c_{\hat{s}}^{12}/p^{21} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$Y^{\setminus}(k)$ is the (k-invariant) matrix of events k payoffs (in states $1, ..., \hat{s}$) and $D\left(\ell_2^{\setminus}\right)$ is the matrix having diagonal coefficients $\left(\ell_2^1, ..., \ell_2^{\hat{s}}\right)$. To conclude the proof, we need to show that $Va^T = 0$ if and only if $a = 0$, i.e. that V has full rank. Use the last block of columns to get rid of all the assets payoffs but inside money and drop the corresponding rows and columns. Subtract the block of columns of $k = 1$ from the blocks $k > 1$ and add the blocks of columns so obtained, for $k = 2, ..., K-1$, to the block relative to $k = 1$. Finally, divide the last column by K and subtract it (multiplied by ℓ_2^s) from the the first block of columns, obtaining

$$\begin{bmatrix} \begin{matrix} T & * & * & * \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 \end{matrix} & -\Upsilon & \cdots & -\Upsilon & \begin{bmatrix} c_I^{11}/p^{11} \\ c_I^{12}/p^{21} \\ \cdots \end{bmatrix} \\ 0 & -I_{\hat{s}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_{\hat{s}} & 0 \\ [0] & [0] & \cdots & [0] & K \end{bmatrix}.$$

Hence, V has full rank provided that the top left (2×2) submatrix T has rank 2, where

$$T = \begin{bmatrix} 1 - \ell_2^1 c_I^{11}/Kp^{11} & -\ell_2^2 c_I^{11}/Kp^{11} \\ -\ell_2^1 c_I^{12}/Kp^{21} & 1 - \ell_2^2 c_I^{12}/Kp^{21} \end{bmatrix}.$$

By direct computation, $detT = \left(K - \ell_2^1 c_I^{11}/p^{11} - \ell_2^2 c_I^{12}/p^{21}\right)/K$.

Fix p^{11}, p^{21}, ℓ_1^1, ℓ_1^2, ℓ_H^1, ℓ_H^2 and K.

Then, remembering the definition of c_I^{11} and c_2^2, one obtains $detT = \left(1 - \left((\ell_1^2 - \ell_H^2)\ell_2^1 + (\ell_H^1 - \ell_1^1)\ell_2^2\right)/(\ell_1^2\ell_H^1 - \ell_1^1\ell_H^2)\right)$.

By construction, either $(\ell_1^2 - \ell_H^2) \neq 0$ or $(\ell_H^1 - \ell_1^1) \neq 0$ or both. If, for instance, the second term is non zero, perturb agent 2's utility function by $\delta_2 = \left[\delta_2^0, 0, \delta_2^2, ..., \delta_H^S\right]$, $\delta_2^2 \neq 0$, and δ_2^s, $s \neq 1, 2$, chosen so that $R(q)\delta_2 = 0$, which can always be done because $\left[y^{k1} \cdots y^{kS}\right]$ is in general position, for each k.

Notice that this last perturbation of the utility function must be applied to some $h \neq 1, H$.

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Growth in Economies With Non Convexities: Sunspots and Lottery Equilibria, Theory and Examples *

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Summary. We investigate the relation between lotteries and sunspot allocations in a dynamic economy where the utility functions are not concave. In an intertemporal competitive economy, the household consumption set is identified with the set of lotteries, while in the intertemporal sunspot economy it is the set of measurable allocations in the given probability space of sunspots. Sunspot intertemporal equilibria whenever they exist are efficient, independently of the sunspot space specification. If feasibility is, at each point in time, a restriction over the average value of the lotteries, competitive equilibrium prices are linear in basic commodities and intertemporal sunspot and competitive equilibria are equivalent. Two models have this feature: Large economies and economies with semi-linear technologies. We provide examples showing that in general, intertemporal competitive equilibrium prices are non-linear in basic commodities and, hence, intertemporal sunspot equilibria do not exist.

The competitive static equilibrium allocations are stationary, intertemporal equilibrium allocations, but the static sunspot equilibria need not to be stationary, intertemporal sunspot equilibria. We construct examples of non-convex economies with indeterminate and Pareto ranked static sunspot equilibrium allocations associated to distinct specifications of the sunspot probability space. Furthermore, we show that there exist large economies with a countable infinity of Pareto ranked static sunspot equilibrium allocations with aggregate pro capita consumption invariant both across equilibria as well as across realizations of the uncertainty. This implies that these equilibria are equilibria of a pure exchange economy with non-convex preferences.

Key words: Lottery Equilibria, Sunspot Equilibria, Non-convexities.
JEL classification numbers: D84, D90.

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1 Introduction

Two Views of Sunspot Economies

The sunspot literature, originated by the Cass and Shell, [3], studied convex economic environments where the First Welfare Theorem does not hold. In these economies the introduction of ”extrinsic uncertainty” may change the set of equilibrium allocations. Indeed the last assertion became a Folk theorem, [16]. Most notably, there are economies where Pareto efficient *certainty* equilibria (i.e., equilibria where sunspots do not matter) coexist with inefficient sunspot equilibria (e.g., [1]). This first research saw sunspot as a socially undesirable, but possible, and to some extent, pervasive, equilibrium outcome of competitive environments.

In the recent years, research on sunspots has taken quite a different route. The objects of study are economies with non-convexities (see for instance [10]). It was soon clear that non convexities in production or in preferences had very different implications.

When non-convexities are present only in the production sets of the firms, the introduction of sunspots is immaterial. Given the linearity of the profit functions, a sunspot equilibrium is an equilibrium of the "concavified" economy and, thus, sunspots do not matter, [2].

The situation is very different when non-convexities are present in the consumers' preferences or consumption set. Most of the literature deals with static economies with non convexities of the consumption set, generated by either indivisibilities, [17], or incentive compatibility constraints, [8]. In these environments randomness in allocations may improve welfare. The latter can be generated in two different ways, either by introducing lotteries or by introducing sunspots. In the sequel, we use the term *competitive equilibrium* to refer to the equilibrium outcome of an environment where individuals face a complete set of markets for lotteries and the term *sunspot equilibrium* when they face, for given specification of the probability space of extrinsic uncertainty, a complete set of (sunspots) contingent markets.

Competitive Equilibria and Sunspot Equilibria

Two major differences between the concept of competitive equilibrium and sunspot equilibria are clear, and both of them are crucial. The first difference concerns the nature of the consumption sets. With the competitive notion, agents may choose any probability distribution over basic consumption bundles, as in [11]. With the sunspot notion, they choose measurable maps with respect to an exogenously given probability space of extrinsic uncertainty $(\Omega, \mathfrak{A}, \hat{\sigma})$.

The second difference concerns the nature of prices in the two concepts. In the lottery economy prices are linear in lotteries and, hence, they can be non-linear in commodities. In the sunspot economy, prices are linear in sunspot contingent commodities. Thus, a first natural question is to investigate whether competitive and sunspot equilibria are "equivalent." There are two aspects to this question. The first is at the level of individual behavior and it concerns the different nature of the consumption sets under the two specifications. The second is at an equilibrium level and it concerns the nature of equilibrium prices. Any individual sunspot allocation induces a probability distribution over consumption bundles and, hence, it induces a lottery. However, the viceversa need not be true. A lottery induces a stochastic allocation, but this may be either not measurable with respect to the given probability space of extrinsic uncertainty, $(\Omega, \mathfrak{A})$, or incompatible with the given sunspot probability law, $\hat{\sigma}$. When each lottery induces a (compatible) sunspot allocation, lotteries and sunspot allocations are equivalent. Garret, Kreister, Qui and Shell [5] show that equivalence is achieved by any "continuous randomizing device"[3]. When the probability space of extrinsic uncertainty is not rich enough, sunspot equilibrium allocations, if they exist, may be Pareto suboptimal. Different specifications of $(\Omega, \mathfrak{A}, \hat{\sigma})$ may yield different equilibria with, in principle, Pareto ranked allocations ([17], [4]). This is not a contradiction of the First Fundamental Theorem of Welfare Economics. Pareto optimality is relative to a given consumption set, while in the previous statements we are comparing allocations in different consumption sets. [4]

However, when the extrinsic uncertainty space is $([0,1), B, L)$, under fairly general assumptions, every sunspot equilibrium allocation induces an equilibrium lottery allocation. If equilibrium allocations of the lottery economy are supported by linear prices, the viceversa is also true, ([5] and [6]). This equivalence result is based on the assumed existence of sunspot equilibria and, equivalently, of lottery equilibria supported by linear prices.

Thus we are led to the second aspect of the problem: singling out environments where competitive equilibrium prices are linear. In any economy where feasibility is a restriction on the average values of (joint) lotteries, equilibrium prices may be restricted (without loss of generality) to be linear. One such environment is a large pure exchange economy (with finitely many types). If each household of type i consumes i.i.d. lotteries β_i, by the law of large numbers, the aggregate average allocation is, with probability 1, deterministic and equal to the average (appropriately taken over types and lotteries) consumption. Feasibility in a large exchange economy requires that average endowment to be equal to average consumption. Hence, feasibility is just a restriction on the average value of the lotteries. Thus, in large economies lottery and sunspot equilibrium allocations are equivalent, [8] and [5].

[3] For instance, the probability space $([0,1), B, L)$, with B denoting the Borel subsets of $[0,1)$ and L the Lesbegue measure.

[4] In this literature as well as in this paper, a feasible (sunspot) allocation is efficient if it does not exists a feasible lottery that Pareto dominates it. Furthermore, it is understood that preferences are extended over random allocations via expected utility.

The Model

We depart from the existing literature in two aspects. First, the lack of convexity is generated by the *preferences* of the households and not by the nature of their consumption set. In previous work, [12], we have studied the Lancaster model of characteristics, [9]. When the household production function transforming commodities into characteristics is concave, but not linear, the derived utility function over commodities (i.e., the composition of the concave utility function over characteristics with the concave production function) can be any continuous function. Furthermore, a long history of experimental evidence indicates the lack of linearity of the household production function. Thus, the Lancaster model of characteristics provides a sound justification for the study of economies with non-convex preferences.

The second key feature of our model is time. We study a continuous time growth economy. In this economy lotteries appear naturally because they are the only sensible way to define the limit of allocation paths (over time) that are in the limit achieving the supremum in the growth problem. Since the utility function is not concave, an optimal deterministic path may not exist: however, the sequence of paths yielding the supremum value is oscillating among extreme points. The natural limit of these paths is a path taking values in lotteries. Hence, in our model, the household consumption set is identified with the set of lotteries and the feasibility requirement equates the effective investment to the average production of investment goods at each instant of time. The definitions of the consumption set and of the feasibility requirement are, in an obvious sense, the outcome of the optimal use of time as a convexifying device. The same type of mechanism is at work for competitive versions of these growth economies. Households facing a budget constraint reproduce, by varying their consumption *over time,* the distribution of their consumption over goods induced by a lottery. In a way they would, by their intertemporal choices, extend the commodity space. This extension is precisely our "natural" choice of commodity space.

A Growth Economy

The growth economy provides thus a natural environment to study the relation between lotteries and sunspot allocations. We examine this relation by using several specifications of the economy, differentiated by time characteristics and the number of households. The competitive equilibrium describes the growth economy where the household consumption set and the feasibility requirement are as previously described and there exist complete markets for time contingent lotteries. In a sunspot equilibrium, for given (time invariant) sunspot probability space $(\Omega, \mathfrak{A}, \hat{\sigma})$, the consumption set of the individuals is identified, at each instant of time, with the set of $(\Omega, \mathfrak{A})$-measurable contingent commodity allocations. Feasibility, conformably with the notion of competitive equilibrium, requires, at each instant of time, the effective investment to be equal to the average production of investment goods. The average, this time, is taken over the realization of the sunspot allocation according to the sunspot probability measure $\hat{\sigma}$.

In a (finite) pure exchange economy with a representative agent, feasibility requires the sunspot allocations to be equal to the sunspot invariant endowments. Thus, sunspots do not matter. However, in the growth economy, the adopted notion of feasibility allows for non trivial randomness of feasible allocations even in the model with a single representative consumer.

An equilibrium can be either intertemporal or static and the economy can be with a single representative household or with a large number of identical ones. The last distinction is immaterial in convex economies, but has important consequences when preferences display non-convexities.

The intertemporal economy is just the standard competitive version of the growth economy. We compare the intertemporal competitive equilibrium allocations with the intertemporal sunspot equilibrium allocations. If, at each competitive equilibrium, lotteries are priced linearly, a quite strong result holds true: competitive allocations and sunspot allocations coincide. Also, without loss of generality, probability adjusted prices are sunspot invariant.

Most importantly and in the contrast with the literature, independently of the sunspot space specification, there cannot exists inefficient sunspot equilibria. The reason is quite simple. If a sunspot probability space does not provide enough randomness to reproduce the desired lotteries, time supplies the additional missing noise. This in turn implies that it cannot exists a solution to the household programming problem or, equivalently, that there cannot be an intertemporal sunspot equilibrium for "inadequate" specifications of $(\Omega, \mathfrak{A}, \hat{\sigma})$. Hence, the key question is to single out environments guaranteeing price linearity of the lotteries in basic commodities. As already argued, the large economy is always one of such environments. The second requires a non-generic restriction on the technology that we call semi-linearity. With a semi-linear technology, for each given vector of capital goods, the isoquant curves of investment and consumption goods are linear (as in the text book growth model). Thus, a lottery is feasible if its average value is feasible. The latter is the key desideratum to obtain linear supporting prices.

However, with a representative household (or with finitely many), for general specifications of the technology, lottery prices are non-linear and sunspot equilibria do not exists. We provide examples of this fact by using the static version of the economy. The notion of static equilibrium is the restriction of the definition of intertemporal equilibrium to vectors of allocations and prices that are constant over time. Thus, each static competitive (lottery) equilibrium is a stationary, intertemporal competitive equilibrium and viceversa.

Stationary allocations are important, simple to analyze and they are constructed, in standard economies, using our notion of static equilibria. Indeed in standard, convex environments, static and stationary intertemporal equilibria coincide. However, in our environment, sunspot (static) allocations behave differently. There are non-convex economies with indeterminate and Pareto ranked static sunspot equilibrium allocations associated with distinct specifications of probability space $(\Omega, \mathfrak{A}, \sigma)$. By what previously said, inefficient static sunspot allocations can never be stationary, intertemporal sunspot equilibrium allocations. Thus, when preferences are not convex, static sunspot equilibria are an inadequate tool to analyze stationary equilibrium allocations of growth economies.

Furthermore, we show that there exist large economies with a countable infinity of Pareto ranked static sunspot equilibrium allocations with aggregate pro capita consumption invariant both across equilibria as well as across realizations of the uncertainty. We select the technology to make sure that these equilibria are equilibria of a large pure exchange economy with non-convex preferences.

2 The Representative Household Economy

We model a growth economy with a representative household. The model is standard except for the lack of convexity of the household's preferences. The particular notions of competitive and sunspot equilibria that we use are much related to the solution of an optimal growth problem with, potentially, a non-concave utility function. Hence, in order to clearly explain the various adopted definitions of equilibrium we turn next (and quickly) to the optimal growth problem. A detailed analysis of the growth problem is in Rustichini and Siconolfi ([13]).

2.1 The Optimal Growth Problem

Consider a growth economy with a representative individual. The consumption set in the basic commodity space is Y, a non-empty, compact and convex subset of $\Re^C_+$. The instantaneous utility function, $U : Y \to \Re$, is continuous, but not necessarily concave. A vector of capital goods is $x \in \Re^K_+$. The continuous and jointly concave function $F : \Re^K_+ \times Y \to \Re^K_+$ describes the technology. Let $\hat{y} : Y \to \Re_+$ denote a measurable map, $\hat{y} = (\hat{y}(t))_{t \geq 0}$. For given initial condition $\hat{x}(0) = x_0$, the optimal growth problem is:

$$\sup_{\hat{y}} \int_0^\infty e^{-rt} U(\hat{y}(t)) dt \tag{1}$$

subject to the constraint:

$$\dot{x}(t) = F(\hat{x}(t), \hat{y}(t)),\ \hat{x}(0) = x_0,\ \hat{y}(t) \in Y. \tag{2}$$

If U is not concave, an optimal solution to (1) may not exist.

The reason is simple. If preferences are not convex, a consumption path (that is, the $\hat{y}$ path) can give, over some time interval, a strictly higher utility than the path equal to the average over time of the original path. So variability over time is, from the point of view of the allocation of consumption, desirable. But the opposite is true from the point of view of production: a variable consumption path can yield a strictly smaller asset accumulation than the path equal to the average over time. Making the variability occur in shorter and shorter time periods can reconcile the two desiderata. The utility is still going to be roughly equal to the average of utilities in different points in time. The loss in production is going to be smaller and smaller because the resulting oscillations in the x path are smaller and smaller. But the limiting measurable optimal path may not exist. The next example illustrates this in a simple economy.

Example 1. An economy with no optimal growth path

The economy is described by $C = K = 1$, and $F(x, y) = f(x) - y$, with f strictly concave and such that $f'(1) = r$ and $f(1) = 1$. The initial condition for the capital stock is $x_0 = 1$.

The utility function U is monotonically increasing, but not concave. There is however a concave function V that satisfies $V(y) = U(y)$ for $y \in [0, 1/2] \cup [3/2, \infty]$, and $V(y) = \lambda U(1/2) + (1 - \lambda)U(3/2) > U(y)$, for each $y = \lambda(1/2) + (1 - \lambda)(3/2)$, $\lambda \in (0, 1)$.

The optimal growth problem when the utility function is V (rather than U) has, given the choice of our initial condition, the unique solution $y(t) = x(t) = 1$, for all t. The value to the problem is $\frac{1}{2}(U(1/2) + U(3/2)) = V(1)$. Furthermore, since $V(y) \geq U(y)$, for all y, the value to the growth problem (1) with the utility function V is greater or equal to the value of the problem with the utility function U.

Consider now the growth problem with U. Fix an arbitrary time interval. Consider the consumption path $\hat{y}$ that alternates period by period the consumption levels 1/2 and 3/2. For r close to zero, the value of this path $U(\hat{y})$ is close to $\frac{1}{2}(U(1/2) + U(3/2)) = V(1)$. Since the production function is strictly concave, variability is costly. Thus, since the average (over time) consumption of the path is 1, $\hat{y}$ is, given the initial condition, not feasible. However, by making the time period arbitrarily small, the cost of the variability converges to zero and the consumption path converges (in the appropriate topology, the narrow topology: see below, and [18] for details) to the time invariant path in the space of probability measures that assigns to both consumption levels, 1/2 and 3/2, a probability 1/2. ■

2.2 The Relaxed Problem

Let $\mathcal{M}_{+,1}(Y)$ denote the set of Borel probability measures on Y. For any probability distribution $\beta \in \mathcal{M}_{+,1}(Y)$, and any real valued function $g(\cdot, \cdot)$, $g(\cdot, \beta)$ denotes the expectation of g according to β, i.e., $g(\cdot, \beta) = \int_Y g(\cdot, y)\beta(dy)$. Let $\hat{\beta} : \Re_+ \to \mathcal{M}_{+,1}(Y)$ be a measurable map. The relaxed (weak) problem associated to 1 is:

$$\sup_{\hat{\beta}} \int_0^\infty U(\hat{\beta}(t))dt \tag{3}$$

subject to the constraint:

$$\dot{x}(t) = F(\hat{x}(t), \hat{\beta}(t)), \hat{x}(0) = x_0, \hat{\beta}(t) \in M_{+,1}(Y) \tag{4}$$

In the programming problem (3), the planner can choose a path of lotteries over the consumption set and the feasibility constraints are written in average. The value of (3) is greater or equal to the value of (1). Furthermore, in (3), the utility function is linear in the control variable, $\beta \in \mathcal{M}_{+,1}(Y)$. Hence, under our specification, there exists an optimal solution $(\hat{x}, \hat{\beta})$ to (3), while (1) may not have an optimal solution. A first known result shows that the supremum of the problem (1) coincides with the value of the relaxed problem (3). Most importantly, a second known result shows that the sequence of deterministic trajectories with value approximating the supremum of (1) converges in an appropriate topology (the narrow topology) to $(\hat{x}, \hat{\beta})$, the optimal solution to (3) (see [18] for details).

These two results justify our characterization of efficient paths by means of the programming problem (3). However, the interpretation should be clear. The opportunity of selecting paths of lotteries over Y as well as the average form of the feasibility requirements are not special assumptions. They emerge from the optimal use of time as a convexifying device.

3 Intertemporal Competitive Equilibrium

The definition of competitive equilibrium is the natural extension of the concept of competitive equilibrium for an economy with complete markets where the household has a non-concave utility function. Faced with a budget constraint, the household will try to randomize consumption to achieve higher utility. If a complete market for lotteries over consumption sets exists, he will be able to do this. The randomization over consumption corresponds to the randomization he can achieve in the dynamic economy by altering consumption over time (*chattering*). The possibility of choosing lotteries over the space of basic consumption goods allows him to randomize at each point in time, rather than *across* time as he does with chattering. Hence, we identify the consumption set, at each t, with $\mathcal{M}_{+,1}(Y)$. The (lottery) price domain is $\mathcal{M}_{+,1}(Y)^*$, the dual of $\mathcal{M}_{+,1}(Y)$. It contains (an isomorphic copy of) the set of continuous functions on Y. We write $< \hat{p}, \hat{\beta} >$ the dual pair. We denote by L the Lebesgue measure on the real line. Firm and household allocations are vectors of paths

$$(\hat{x}_f, \hat{x}_h, \hat{a}_f, \hat{a}_h, \hat{\beta}_f, \hat{\beta}_h)$$

where, $\hat{x}_f(t)$ and $\hat{x}_h(t) \in \Re^K$ are, respectively, the supply and the demand of capital , $\hat{a}_f(t)$ and $\hat{a}_h(t) \in \Re^K$ are demand and supply of investment, and $\hat{\beta}_h(t)$ and $\hat{\beta}_f(t) \in \mathcal{M}_{+,1}(Y)$ are supply and demand of (lotteries over) consumption allocations. Prices are vectors of paths:

$$(\hat{q}, \hat{b}, \hat{p})$$

where $\hat{b}(t) \in \Re^K$ is the rental price of capital, $\hat{q}(t)$ is the price of investment good, and $\hat{p}(t) \in \mathcal{M}_{+,1}(Y)^*$ is the price of consumption allocations. The space of allocations is a subset of the space of measurable functions from $\Re_+$ to $\Re^{2K} \times \mathcal{M}_{+,1}(Y)$. The precise space is the set of measurable functions, $(\hat{x}, \hat{a}, \hat{\beta})$, such that the discounted L_1 norm is finite. Prices are linear functional on this space, hence measurable functions with finite, discounted essential sup norm. The inner product is defined in the natural way:

$$< (\hat{q}, \hat{b}, \hat{p})(\hat{x}, \hat{a}, \hat{\beta})) >\equiv \int_0^{+\infty} [\hat{q}(t)\hat{x}(t) + \hat{b}(t)\hat{a}(t) + < \hat{p}(t), \hat{\beta}(t) >]dt.$$

In the intertemporal economy, the firm and the household solve the following problems.

Definition 1. *For a given vector of prices, the firm's problem is:*

$$\max_{(x,a,\gamma)} \int_0^{+\infty} [-\hat{b}(t)\hat{x}(t) + < \hat{p}(t), \hat{\gamma}(t) > + \hat{q}(t)\hat{a}(t)]dt \qquad (5)$$

subject to

$$\hat{a}(t) = F(\hat{x}(t), \hat{\gamma}(t)) \ L - \textit{almost every } t; \qquad (6)$$

Let $v(\hat{q}, \hat{b}, \hat{p})$ denote the value of the firm, i.e., $v(\hat{q}, \hat{b}, \hat{p}) = \int_0^{+\infty} [-\hat{b}(t)\hat{x}(t) + < \hat{p}(t), \hat{\gamma}(t) > + \hat{q}(t)\hat{a}(t)]dt$, where $(\hat{x}, \hat{a}, \hat{\beta})$ is a solution of the firm's problem (5) at $(\hat{q}, \hat{b}, \hat{p})$;

Definition 2. *For a given vector of prices and value of the firm $v(\cdot)$, the consumer's problem is:*

$$\max_{(x,a,\gamma)(t)} \int_0^{+\infty} e^{-rt} U(\hat{\gamma}(t))dt, \qquad (7)$$

$$\textit{subject to } \int_0^{+\infty} < \hat{p}(t), \hat{\gamma}(t) > dt - \int_0^{+\infty} [\hat{b}(t)\hat{x}(t) - \hat{q}(t)\hat{a}(t)]dt \leq v(\hat{q}, \hat{b}, \hat{p});$$

and

$$\hat{x}(0) = x_0, \ \dot{x} = \hat{a}(t), \ L - a.e.$$

Although U may be a non-concave function of $y \in Y$, U, and, therefore, $\int_0^{+\infty} e^{-rt} U(\cdot)dt$, is a linear function of the lotteries, $\gamma \in M_{1,+}(Y)$. Furthermore, if at equilibrium, preferences satisfy local non satiation, the path $(\hat{x}_h, \hat{a}_h)$ is chosen to maximize the wealth of the household. Wealth maximization is independent of the particular shape U and it is always a linear problem. Hence, (7) is a concave programming problem. The nature of the investment at time t, $\dot{x}(t)$ requires some clarification. Investment is a random variable, dependent on the realization of the random variable input. However, since this quantity enters linearly into the consumer's problem, we may assume that the consumer chooses a deterministic investment at each time. The market clearing condition on investment is satisfied L- almost surely because the market clearing condition on the stock of capital in each period is satisfied. If we write the market clearing condition explicitly, it is stated as an equality between the expected value of demand and supply of investment. We identify an economy with an array (U, F, x_0).

Definition 3. *A competitive equilibrium of the economy (U, F, x_0) is a vector of prices and allocations*

$$(\hat{q}, \hat{b}, \hat{p}; \hat{x}_f, \hat{x}_h, \hat{a}_f, \hat{a}_h, \hat{\beta}_f, \hat{\beta}_h)$$

such that

1. *$(\hat{x}_f, \hat{x}_h, \hat{a}_f)$ is a solution of the firm's problem (5);*
2. *$(\hat{x}_h, \hat{a}_h, \hat{\beta}_h)$ is a solution of the consumer's problem (7);*
3. *Markets clear:*

$$\hat{\beta}_f(t) = \hat{\beta}_h(t),\ \hat{x}_f(t) = \hat{x}_h(t),\ \hat{a}_f(t) = \hat{a}_h(t),\ L - \textit{almost every } t;$$

In the framework we have described, the economy is a neoclassical economy with concave (linear) preferences and convex technology. Hence under technical assumptions a competitive equilibrium exists, is efficient, and the second welfare theorem holds.

3.1 Static Equilibrium

The notion of static equilibrium is the restriction of the definition of intertemporal equilibrium to vectors of allocations and prices that are constant over time.

Definition 4. *A static equilibrium is a vector*

$$(q, a, p; x_f, a_f, \beta_f, \beta_h)$$

where $x \in R^K$, $\beta_f, \beta_h \in \mathcal{M}_{+,1}(Y)$; $p \in \mathcal{M}_{+,1}(Y)^*$, $q \in \Re^K$, and $v \in R$ such that

1. *β_h is a solution of the consumer's problem:*

$$\max_{\gamma \in \mathcal{M}_{+,1}(Y)} U(\gamma),\ \textit{subject to } < p, \gamma > \leq v; \tag{8}$$

2. The firm's value gross of the capital expenses is

$$v =< p, \beta_f > + qa'$$

3. (x_f, a_f, β_f) *is a solution of the firm's problem:*

$$\max_{(x',\beta)} -rqx' + [< p, \beta > + qa'] \text{ subject to } a' \leq F(x', \beta) \tag{9}$$

4. markets clear:

$$F(x, \beta_f) = 0,\ \beta_h = \beta_f,\ a_f = 0$$

The relation between the static and the intertemporal equilibrium notions is fairly obvious. A stationary, intertemporal equilibrium is a static equilibrium. Viceversa, a static equilibrium (x, β, p, q) is a stationary equilibrium of the intertemporal economy, i.e., given the initial condition $x(0) = x$, the path, $(\hat{q}, \hat{b}, \hat{p}, \hat{x}, \hat{a}, \hat{\beta})(t) = (x, 0, \beta, e^{-rt}(q, 0, p))$, for all t, is an equilibrium of the intertemporal economy. Hence, the stationary allocation (x, β) is an efficient allocation of the intertemporal economy. A static allocation (x, β) is efficient if it is an efficient stationary allocation of the intertemporal economy, i.e., if there exists a $q \in \Re^K$ such that $(x, \beta) \in \arg\max U(\beta) + qF(x, \beta)$. (see, [13]).

4 Static and Intertemporal Sunspot Equilibria

In agreement with the definitions of competitive equilibria, we give intertemporal and static definitions of sunspot equilibria. The common element is the existence of an exogenous (and, for sake of simplicity, time invariant) standard measurable space of extrinsic uncertainty (sunspots) $(\Omega, \mathcal{A})$ together with an exogenous probability, $\hat{\sigma}$. The definitions of sunspot equilibria is a simple restatement of the definitions of competitive equilibria in an environment where there is a complete set of contingent (on the realization $\omega \in \Omega$) commodities, but where lotteries are absent. Capital and investment good prices are restricted to be sunspot invariant. As already discussed, this is a consequence (given the sunspot invariance of the prices q) of the concavity in $(\hat{x}, \hat{a})$ of both the firm and the household programming problem. Let $\mathcal{B}_{\Re_+}$ denote the Borel sets of $\Re_+$. In the intertemporal economy, the space of allocations is the space of paths of $(\Omega \times \Re_+, \mathcal{A} \times \mathcal{B}_{\Re_+})-$measurable functions $(\hat{x}, \hat{a}, \hat{y}) : \Omega \times \Re_+ \to \Re_+^{2K} \times Y$ satisfying the restriction "$(\hat{x}, \hat{a})(\omega, t)$ is $\omega-$invariant", with norm $\| (\hat{x}, \hat{y}) \|_{1,r} \equiv \int_0^{+\infty} e^{-rt} \| \hat{x}(t), \sup_{\omega \in \Omega} \hat{y}(\omega, t) \| dt$, finite. Prices are linear functional on this space, hence measurable functions with finite norm $\| (\hat{q}, \hat{b}, \hat{p}) \|_{\infty,r} \equiv \sup_{(t) \in R_+} e^{rt} \int_\Omega \| (\hat{q}, \hat{b}, \hat{p}(\omega))(t) \| \sigma(d\omega)$. Hence, $\hat{p}(\omega, t)$ is an element of $\Re_+^L$ for each (ω, t), and, thus, prices are linear (in contingent commodities). In the static economy, the space of allocations is the space of $(\Omega, \mathcal{A})-$measurable functions $(\hat{x}, \hat{a}, \hat{y}) : \Omega_+ \to \Re_+^{2K} \times Y$ satisfying the restriction that "$(\hat{x}, \hat{a})(\omega)$ is $\omega-$invariant. ", while prices are linear functionals over this space. As for the notion of competitive equilibrium, in a sunspot equilibrium, the equivalent constraint of equation 6 in the definition 1 reads

$$\hat{a}(t) = \int_{\Omega} F(\hat{x}(t), \hat{y}(\omega, t))\hat{\sigma}(d\omega)\ L - \text{almost every } t; \tag{10}$$

while for a static sunspot equilibrium, feasibility requires, in agreement with the definition of a static sunspot equilibrium:

$$0 = \int_{\Omega} F(x, \hat{y}(\omega)\hat{\sigma}(d\omega)\ L - \text{almost every } t; \tag{11}$$

We do not explicitly write down the definitions of static and intertemporal sunspot equilibria because, modulo the change in the allocation sets, are identical to the definitions of intertemporal and static competitive equilibria.

When, judging the efficiency of sunspot allocations we use the following definition of efficiency:

Definition 5. *A feasible sunspot allocation* $(\hat{x}, \hat{a}, \hat{y})$ *of the intertemporal economy* $(U, F, x_0, (\Omega, \mathcal{A}, \hat{\sigma}))$*, is efficient if it does not exists a feasible (lottery) allocation* $(\hat{x}', \hat{a}', \hat{\beta}')$ *of the economy* (U, F, x_0) *such that*

$$\int e^{-rt}(\int_{\Omega} U(\hat{y}(\omega, t))\hat{\sigma}(d\omega))dt < \int e^{-rt}(U(\hat{\beta}'(t))dt.$$

5 Sunspot and Lottery Equilibrium Allocations

5.1 Sunspot Equilibria in Finite Economies

As already explained, two major differences between the concept of competitive equilibrium and sunspot equilibria are clear and crucial. The first difference concerns the nature of the consumption sets, while the second concerns the nature of prices. We start by analyzing the first difference.

Sunspot and Lottery Allocations

In economies with non convexities, there are two aspects to the link between lottery and sunspot allocations. 1) The space of sunspots $(\Omega, \mathcal{A}, \hat{\sigma})$ must be rich enough to provide the randomness necessary to reproduce lotteries. 2) Since $(\Omega, \mathcal{A}, \hat{\sigma})$ is the common randomization device, individual sunspot allocations must be appropriately correlated to be , $\hat{\sigma} - a.e.\omega$, feasible. This second aspect is obviously absent in our representative economy. Later, when we study the large economy, 2) will manifest necessarily itself. The first aspect is however present and it needs to be clarified.

In finite, complete markets economies different specifications of the sunspot space may yield different and Pareto ranked sunspot equilibrium allocations

([17], [4]). In section 8, we work out an example of a large economy with non convex preferences, which may describe a static economy as well a pure exchange one, where distinct specifications of the space of sunspot yield a countable infinity of Pareto ranked sunspot allocations. The existence of multiple sunspot equilibria provides the rationale for the search of ways to refine sunspot equilibria ([7], [6]).

We identify the set of sunspots with the interval $[0,1)$, their probability with the Lebesgue measure L and we allow for different specifications of the $\sigma-$algebra, $\hat{B}$, with $\hat{B} \subset B$, the Borel sets over the interval $[0,1)$. As shown in [5], $([0,1), B, L)$ (or any probability space isomorphic to it) accomplishes desideratum 1), in the following precise sense: Let C be a Borel subset of Y and observe that $\hat{y}^{-1}(C)$ is a Borel subset of $[0,1)$, for $\hat{y} \in Y([0,1), B)$, the set of $([0,1], B) - measurable$ functions from $[0,1)$ to Y. A lottery $\beta \in M_{1,+}(Y)$ and an allocation $\hat{y} \in Y([0,1), B)$ are equivalent, $\hat{y} \sim \beta$, if for any C :

$$\beta(C) = L(\hat{y}^{-1}(C)) = L\{\omega : \hat{y}(\omega) \in C\}, \text{ and, therefore, } \beta = L \circ \hat{y}^{-1}.$$

[5], in Lemma 4, proves that for each $\beta \in M_{1,+}(Y)$, there are $\hat{y} \in Y([0,1), B)$ such that $\hat{y} \sim \beta$ and viceversa. Also, by the same argument, for each given sunspot space $(\Omega, \mathcal{A}, \hat{\sigma})$ and each $y \in Y(\Omega, \mathcal{A})$ there exists $\hat{y}^* \in Y([0,1), B, L)$ such that $\hat{y}^* \sim \hat{y}$. Hence, $\{[0,1), B, L\}$ is rich enough to reconstruct lotteries as well as any sunspot allocation measurable with respect to any arbitrarily given standard probability space $(\Omega, \mathcal{A}, \hat{\sigma})$. From now on, we identify a sunspot economy with the array $(U, F, x_0, \hat{B})$, $\hat{B} \subset B$. While, as already stated, a lottery economy is an array (U, F, x_0).

Constant Price Sunspot Equilibria

Again following [5], a key role in our analysis is played by a particular notion of sunspot equilibria, that we call constant prices sunspot equilibria, hereafter, CPSE. An intertemporal CPSE is an intertemporal sunspot equilibrium with $(\Omega, \mathcal{A}, \hat{\sigma}) = ([0,1), B, L)$ and $\hat{p}(t, \omega) = p(t)$, for all $\omega \in [0,1)$ and some $p(t) \in \mathfrak{R}^L$. The natural adaptation of the last definition to the static environment provides the definition of CPSE for the static economy. Whenever we talk about a CPSE of some economy, it is understood that $\hat{B} = B$ so that we can talk of a CPSE of the economy (U, F, x_0) without ambiguity.

In pure exchange economies with non-convexities, CPSE have a very powerful property. Under fairly general conditions, these two results hold true. 1) To each lottery equilibrium supported by linear prices it corresponds an equivalent CPSE and viceversa ([5], Theorem 3). 2) The sunspot invariance price restriction is without loss of generality.

Thus, in finite economies, whenever CPSE exist, they perform as well as linearly priced lotteries. However, the existence of CPSE (or, in general, of any sunspot equilibrium) is related to the form of the feasibility constraint. If the latter is a restriction on the average (sunspot or lottery) allocation (as in a large economy), CPSE exist, otherwise, their existence as well as the existence of any sunspot allocation may fail. Equivalently, as we will clarify later, CPSE exists if competitive (lottery) equilibria have linear supporting prices, otherwise they do not. The example that follows illustrates this simple point.

Example 2. A one period economy without sunspot equilibria.

We consider a one period, two commodities, pure exchange economy with a unique individual. The feasibility restricts the realizations of lotteries and sunspot allocations (as opposite to their average). In this economy, the unique efficient allocation can be supported in the lottery economy, but not in the sunspot one.

The consumer utility function is $U(y) = \sqrt[g]{(y_1)^2 + (y_2)^2}$, $g > 0$, and the endowments are $e = (1, 1)$. For all g, the unique efficient allocation coincides with e.

In the economy with lotteries, for given prices $p \in \mathcal{M}_{+,1}(Y)^*$, the consumer seeks a lottery $\beta \in \arg\{\max_{\beta \in M_{+,1}(Y)} U(\beta)$ subject to $\int_{y \in Y} p(y)\beta(dy) \leq p(e)\}$, while feasibility restricts lotteries into the set $\{\beta \in \mathcal{M}_{+,1}(Y) : \beta\{y \in Y : y \leq e\}\} = 1$. The efficient lottery allocation is δ_e (i.e., the degenerate lottery that assigns probability 1 to e) supported by, for instance, the price $p(y) = U(y) \in \mathcal{M}_{+,1}(Y)^*$.

In the economy with sunspot space $([0, 1), B, L)$, we restrict, without loss of generality, prices to be $\omega -$ *invariant* and thus, for given $p \in \Re^2_{++}$ the individual seeks a sunspot allocation $\hat{y} \in \arg\{\max \int U(\hat{y}(\omega))d\omega$ subject to $\int p(\hat{y}(\omega) - e)d\omega \leq 0\}$. Feasibility restricts the sunspot allocations in the set $\{\hat{y} : L\{\hat{y} : \hat{y}(\omega) \leq e\} = 1$. If $g \geq 2$, the optimal solutions are, for each ω, the consumption bundle $(\frac{p_1+p_2}{p_1}, 0)$, if $p_1 < p_2$, $(0, \frac{p_1+p_2}{p_2})$, if $p_2 < p_1$, or the pair $(2, 0)$ and $(0, 2)$, if $p_1 = p_2$. Thus a CPSE (and more generally, any sunspot equilibrium) does not exist.

Furthermore, in the sunspot economy, for $g \in (0, 2)$, there is not a solution to the consumer problem. To make this point, suppose, without loss of generality, that $p_1 \leq p_2$ and $pe = 1$. Pick $\varepsilon > 0$, set $\hat{y}(\sigma) = 0$, for $\sigma > \varepsilon$ and $\hat{y}(\sigma) = (1/p_1, 0)$, for $\sigma \in [0, \varepsilon)$. Then, $\int U(\hat{y}(\omega))d\omega = (1/p_1)^{2/g}(\varepsilon)^{\frac{g-2}{g}}$ which diverges to infinity as $\varepsilon \to 0$.

In the same economy, with a large number of identical individuals, sunspot equilibria exist and are efficient. The reason is simple. In the large lottery pure exchange economy, by the law of large numbers, feasibility restricts the average lottery consumption, i.e., $\int_Y (y - e)\beta(dy) \leq 0$. Apparently, the situation is different for the sunspot economy, where feasibility restricts, realization by realization, the average taken across individuals of the sunspot allocation. However, the sunspot allocation can be appropriately correlated across individuals in order to make the average taken across individuals equal to the average taken across sunspot realizations. This point is clarified in sections 7 and 8. ∎

There are two important differences between the growth economy and the finite economy that change some aspects of the relation between lotteries and sunspot. To these topics we turn next.

5.2 Intertemporal Sunspot Equilibria

As already said, in the growth economy, in addition to the probability space $([0,1), \hat{B}, L)$, the representative household can use time as a convexifying device. The latter has two implications. First and in contrast to the finite economy results, intertemporal sunspot equilibria of the economy $(U, F, x_0, \hat{B})$ are efficient (in the sense of the definition (5)) for any $\hat{B} \subset B$. Second, all intertemporal sunspot equilibria are, without loss of generality, CPSE. To avoid confusions, we are not showing that sunspot equilibria exist, but rather that, when they exist, they are efficient and CPSE. The first property implies that $([0,1), \hat{B})$ provides, at each t, enough variability to reproduce the optimal lottery and that the latter, by definition of sunspot, can be supported by linear prices. This second property has a key implication. When the representative agents maximize facing linear prices, they optimally select allocations yielding the value of the concave regularizations of their objective functions. The next Lemma makes precise this statement. Let $V = \text{ co } U$ be the concave regularization of the utility function U, since Y is compact, V can be defined as:

$$V(y) \equiv \max_{\beta \in M_{1,+}(Y)}\{U(\beta), \text{ subject to } y(\beta) = y\}, \text{ for } y \in Y.$$

For given $\hat{B} \subset B$, household wealth W and price trajectory $\hat{p}$ consider the two following programming problems:

$$(U) \qquad \max_{\hat{y}(t) \in Y([0,1),\hat{B})} \int_{\Re_+} e^{-rt}\Big(\int_{[0,1)} (U(\hat{y}(\omega,t))d\omega\Big)dt,$$

$$\textit{subject} \text{ to} \int_{\Re_+} \Big(\int_{[0,1)} \hat{p}(\omega,t)\hat{y}(\omega,t)d\omega\Big)dt \leq W$$

and

$$(V) \qquad \max_{\hat{y}(t) \in Y([0,1),\hat{B})} \int_{\Re_+} e^{-rt}\Big(\int_{[0,1)} (V(\hat{y}(\omega,t))d\omega\Big)dt,$$

$$\text{subject to} \int_{\Re_+} \Big(\int_{[0,1)} \hat{p}(\omega,t)\hat{y}(\omega,t))d\omega\Big)dt \leq W.$$

The next lemma explains the relations between the optimal solutions to (U) and (V). Its proof is deferred to the appendix.

Lemma 1. *Any optimal solution to* (U), $\hat{y}^*$, *is an optimal solution to* (V). *Furthermore,*

$$\int_{\Re_+} e^{-rt}\Big(\int_{[0,1)} (U(\hat{y}^*(\omega,t))d\omega\Big)dt = \int_{\Re_+} e^{-rt}\Big(\int_{[0,1)} (V(\hat{y}^*(\omega,t))d\omega\Big)dt.$$

We now exploit Lemma 1 to show that each intertemporal sunspot equilibrium of (U, F, x_0, B) is a sunspot equilibrium of (V, F, x_0, B). To simplify notation we denote by $\hat{\pi}$ prices and by $\hat{\xi}$ allocations.

Proposition 1. *Let* $(\hat{\pi}^*, \hat{\xi}^*)$ *be an intertemporal sunspot equilibrium of the economy* $(U, F, x_0, \hat{B})$*. Then,* $(\hat{\pi}^*, \hat{\xi}^*)$ *is a sunspot equilibrium of the economy* $(V, F, x_0, \hat{B})$*. Furthermore,*

$$\int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{y}^*(\omega, t))d\omega)dt = \int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}^*(\omega, t))d\omega)dt.$$

Proof. The economies $(V, F, x_0; \hat{B})$ and $(U, F, x_0, \hat{B})$, when facing identical price trajectories, have identical profit and household's wealth maximization problems. Thus, we just need to prove that $\hat{y}^*$ is an optimal solution to

$$\max_{\hat{y}(t) \in Y([0,1), \hat{B})} \int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}(\omega, t))d\omega)dt$$

subject to

$$\int_{\Re_+} (\int_{[0,1)} \hat{p}(\omega, t)\hat{y}(\omega, t))d\omega)dt \leq \int_{\Re_+} (\int_{[0,1)} \hat{p}(\omega, t)\hat{y}^*(\omega, t)d\omega)dt.$$

This follows immediately from Lemma 1 which implies as well the second part of the proposition.

The economy $(V, F, r; \hat{B})$ is a concave economy. Therefore, sunspot do not matter. This is the key property that we exploit in order to show both that sunspot equilibria are efficient and that they are, without loss of generality, CPSE.

Proposition 2. *Let* $(\hat{\pi}, \hat{\xi})$ *be a sunspot equilibrium of the intertemporal economy* $(U, F, x_0, \hat{B})$*. Then: 1)* $\hat{\xi}$ *is an efficient allocation of the economies* (U, F, x_0) *and* (V, F, x_0)*; 2)* $(\hat{\pi}, \hat{\xi})$ *is, without loss of generality, a CPSE of the economies* (U, F, x_0) *and* (V, F, x_0)*.*

Proof. By proposition 1, $(\hat{\pi}, \hat{\xi})$ is a sunspot equilibrium of $(V, F, x_0, \hat{B})$. Since the latter is a concave economy, by the First Fundamental Theorem of Welfare Economics, $\hat{\xi}$ is an efficient allocation (according to definition 5) of the economy (V, F, x_0). Suppose that, by contradiction, there exists a feasible lottery allocation $(\hat{x}^*, \hat{a}^*, \hat{\beta}^*)$ of the economy (U, F, x_0) such that:

$$\int_{\Re_+} e^{-rt}U(\hat{\beta}^*(t))dt > \int_{\Re_+} e^{-rt}(\int_{\Omega} (U(\hat{y}(\omega, t))d\omega)dt \tag{12}$$

Since, by the definition $V(y) \geq U(y)$, for all $y \in Y$, and since, by proposition 1, $\int_{\Re_+} e^{-rt}(\int_{\Omega}(U(\hat{y}(\omega, t))d\omega)dt = \int_{\Re_+} e^{-rt}(\int_{\Omega}(V(\hat{y}(\omega, t))d\omega)dt$, 12 implies that $\int_{\Re_+} e^{-rt}V(\beta^*(t))dt > \int_{\Re_+} e^{-rt}(\int_{\Omega}(V(\hat{y}(\omega, t))d\omega)dt$. A contradiction. Thus, 2) holds true. Since $\hat{\xi}$ is efficient for (V, F, x_0) and the latter is a concave economy, the following equalities hold true:

$$\int_{[0,1)} V(\hat{y}(\omega,t))d\omega = V(\int_{[0,1)} \hat{y}(\omega,t))d\omega), \text{ and} \tag{13}$$

$$\int_{[0,1)} F(\hat{x}(t),\hat{y}(\omega,t))d\omega = F(\int_{[0,1)}(\hat{x}(t),\hat{y}(\omega,t))d\omega)).$$

Thus, by 13, $(\hat{x},\hat{a},\bar{y})$ is an efficient allocation of the intertemporal economy (V,F,x_0), for $\bar{y}(t) = \int_{[0,1)} \hat{y}(\omega,t)d\omega$. Therefore, there exists a sunspot invariant price $(\hat{q}^*,\hat{b}^*,\hat{p}^*)$ that supports $(\hat{x},\hat{a},\bar{y})$ as a (degenerate) sunspot equilibrium of the economy (V,F,x_0,B^*), for any $B^* \subset B$. However, again by 13, $(\hat{q}^*,\hat{b}^*,\hat{p}^*)$ supports also the allocation $(\hat{x},\hat{a},\hat{y})$ as an intertemporal sunspot equilibrium of the economy (V,F,x_0,B^*), for any $\hat{B} \subset B^* \subset B$. Hence, to conclude the argument we need to show that $(\hat{q}^*,\hat{b}^*,\hat{p}^*)$ supports $(\hat{x},\hat{a},\hat{y})$ as an intertemporal sunspot equilibrium of the economy (U,F,x_0,B^*), for any $\hat{B} \subset B^* \subset B$.

Since the economies $(V,F,r;B^*)$ and $(U,F,r;B^*)$, when facing identical price trajectories, have identical profit and household's wealth maximization problems, it suffices to show that $\hat{y}(t)_{t\geq 0}$ is an optimal solution to

$$\max_{\hat{y}"(t)\in Y([0,1),B^*)} \int_{\Re_+} e^{-rt}(\int_{[0,1)}(U(\hat{y}"(\omega,t))d\omega))dt$$

$$\textit{subject to} \int_{\Re_+} \hat{p}^*(t)(\int_{\Omega} \hat{y}"(\omega,t))d\omega)dt \leq \int_{\Re_+} \hat{p}^*(t)(\int_{[0,1)} \hat{y}(\omega,t))d\omega)dt$$

Suppose by contradiction that there exists a budget feasible allocation $\hat{y}'$ such that $\hat{y}'(t) \in Y([0,1),B^*)$, for all t, and

$$\int_{\Re_+} e^{-rt}(\int_{[0,1)}(U(\hat{y}'(\omega,t))d\omega)dt > \int_{\Re_+} e^{-rt}(\int_{[0,1)}(U(\hat{y}(\omega,t))d\omega))dt$$

$$= \int_{\Re_+} e^{-rt}(\int_{[0,1)}(V(\hat{y}(\omega,t))d\omega)dt.$$

However, the definition of V and the previous inequality imply that

$$\int_{\Re_+} e^{-rt}(\int_{[0,1)}(V(\hat{y}'(\omega,t))d\omega)dt > \int_{\Re_+} e^{-rt}(\int_{[0,1)}(V(\hat{y}(\omega,t))d\omega)dt.$$

A contradiction.

Thus, by the last proposition, if sunspot equilibria exist, they are "equivalent" to competitive equilibria. As we stated in the introduction, there are two economic environments, where this equivalence is guaranteed: the first is defined by a restriction on the technology F, that we call it "semi-linearity", and the second is the large economy. Next we turn to semi-linear economies.

5.3 Sunspot Equilibria and Semi Linear Economies

An economy (U,F,r) is semi-linear if $F(x,y) = f(x) + By$ for some (concave) function $f : R^K_+ \to R^K$ and matrix B of dimension $L \times K$. Semi-linear economies have an appealing feature: equilibrium prices are (without loss of generality) linear, i.e., $< p, \beta >= p \int y\beta(dy)$, for some $p \in R^L_+$. The latter is obviously a necessary feature to allow for the existence of sunspot equilibria. First we show that prices are indeed linear in a semi-linear economy and than we exploit this feature to construct equivalent CPSE. The next proposition shows that lottery equilibrium allocations of a semi-linear economy are closely related to the equilibrium allocations of the concavified version of the economy. Most importantly, the supporting prices are identical and, therefore, linear. Let $(\hat{a}^*, \hat{x}^*, \hat{y}^*)$ be a feasible allocation of the semi-linear economy (V, F, x_0) and let $(\hat{a}, \hat{x}, \hat{\beta})$ be an allocation of the semi-linear economy (U, F, x_0). The two allocations are equivalent if $(\hat{a}^*, \hat{x}^*)(t) = (\hat{a}, \hat{x})(t)$, $U(\hat{\beta}(t)) = V(\hat{y}^*(t))$ and $y(\hat{\beta}(t)) = \hat{y}^*(t)$, for $L - a.e.t.$

Proposition 3. *Let $F(\cdot) = f(\cdot) + By$. The economies (U, F, r) and (V, F, r) have equivalent efficient allocations. Furthermore, without loss of generality, the supporting prices are in both economies equal and linear.*

Proof. Let $(\hat{a}, \hat{x}, \hat{y})$ be an efficient allocation of the semi-linear economy defined by (V, F, x_0). Let $\hat{\beta}(t) \in \arg\{\max_{\gamma \in M_{1,+}} U(\gamma)$ subject to $y(\gamma) = \hat{y}(t)\}$. By the definition of least concave, $V(\hat{y}(t)) = U(\hat{\beta}(t))$ and by the semi-linearity of the technology, $(\hat{a}, \hat{x}, \hat{\beta})$ is feasible. Thus, the allocations $(\hat{a}, \hat{x}, \hat{y})$ and $(\hat{a}^*, \hat{x}^*, \hat{\beta})$ are equivalent. Since, $(\hat{a}, \hat{x}, \hat{y})$ is efficient for (V, F, x_0), the definition of V implies that $(\hat{a}, \hat{x}, \hat{\beta})$ is efficient for (U, F, x_0). Conversely, let $(\hat{a}, \hat{x}, \hat{\beta})$ be an allocation of the semi-linear economy (U, F, x_0). Then, by the semi-linearity of the technology, $\hat{\beta}(t) \in \arg\{\max U(\gamma)$ subject to $y(\gamma) = y(\hat{\beta}(t))\}$. Thus, $U(\hat{\beta}(t)) = V(y(\hat{\beta}(t))$. Then, the allocations $(\hat{a}, \hat{x}, \hat{\beta})$ and $(\hat{a}, \hat{x}, \hat{y}_{\hat{\beta}})$, $\hat{y}_{\hat{\beta}}(t) = y(\hat{\beta}(t))$, for all t, are equivalent and, quite obviously, $(\hat{a}, \hat{x}, \hat{y}_{\hat{\beta}})$ is an efficient allocation of (V, F, x_0). Thus the first part of the proposition holds true. Consider a pair of efficient and equivalent allocations $(\hat{a}, \hat{x}, \hat{\beta})$ and $(\hat{a}, \hat{x}, \hat{y})$. By the concavity of (V, F, x_0), there exists a price $(\hat{q}, \hat{b}, \hat{p})$, with $\hat{p}(t) \in R^L$, for all t, supporting $(\hat{a}, \hat{x}, \hat{y})$ as a competitive equilibrium allocation. Thus, we need to show that $(\hat{q}, \hat{b}, \hat{p})$ supports $(\hat{a}, \hat{x}, \hat{\beta})$ as an equilibrium allocation of the economy (U, F, x_0). Since the firm profit and the household wealth maximization problems are identical in the two economies, we just have to show that $\hat{\beta}$ is an optimal solution to: $\max \int_{\Re_+} e^{-rt} U(\hat{\gamma}(t)dt$ subject to $\int_{\Re_+} \hat{p}(t) y(\hat{\gamma}(t))dt \leq \int_{\Re_+} \hat{p}(t)\hat{y}(t))dt$ Once again, the definition of V implies the claim.

The proposition allows for an immediate characterization of intertemporal equilibria of the semi-linear economies in terms of CPSE sunspot equilibria.

Corollary 1. *Let $(\hat{q}, \hat{b}, \hat{p}, \hat{x}, \hat{a}, \hat{\beta})$ be a competitive equilibrium with linear prices of the semi-linear economy (U, F, r). Then, there exists a consumption sunspot allocation $\hat{y}$, with $\hat{y}(t) \sim \hat{\beta}(t)$, for all t, such that the vector $(\hat{q}, \hat{b}, \hat{p}, \hat{x}, \hat{a}, \hat{y})$ is an intertemporal CPSE.*

Proof. This is an immediate consequence of the equivalence relation $\sim$. It suffices to observe that, for any given measurable function g and any allocation $\hat{y} \in Y([0,1), B)$, by the standard change of variable formula, $\int_{[0,1)} g(\hat{y}(\omega))\lambda(d\omega) = \int_Y g(y) L \circ \hat{y}^{-1}(dy)$. However, for $\hat{y} \sim \beta$, $\beta = L \circ \hat{y}^{-1}$, hence $\int_Y g(y)\lambda \circ y^{-1}(dy) = \int_Y g(y)\beta(dy)$. It follows that at common prices $(\hat{q}, \hat{b}, \hat{p})$, equivalent paths $(\hat{x}, \hat{a}, \hat{\beta})$ and $(\hat{x}, \hat{a}, \hat{y})$ yield the same value of profits, overall utility and budget constraints. Since for each $\beta \in M_{1,+}(Y)$ there exists equivalent $\hat{y} \in Y([0,1), B)$, and viceversa, the latter implies the claim.

Proposition 3 implies that every competitive equilibrium can be supported by linear prices. Thus, Corollary 1 implies that every competitive equilibrium allocation can be supported as a sunspot equilibrium allocation. Furthermore, Proposition 2, imply that also of the converse of the claim in Corollary 1 holds true. Hence, lotteries and sunspot are in the semi-linear environment equivalent. Propositions 2 has another interesting implication. Consider $(\hat{x}, \hat{a}, \hat{y})$ an inter-temporal sunspot equilibrium allocation of the economy $(U, F, x_0, \hat{B})$, with F (not necessarily a semi-linear). By Proposition 2 , $(\hat{x}, \hat{a}, \hat{y})$ is an inter temporal sunspot equilibrium allocation of $(V, F, x_0, \hat{B})$. Hence, $\int_{[0,1)} V(\hat{y}(\omega, t)d\omega = V(\int_{[0,1)} \hat{y}(\omega, t)d\omega)$ and $\int_{\{0,1)} F(\hat{x}, \hat{y}(\omega))d\omega = F(\hat{x}, \int_{[0,1)} \hat{y}(\omega)d\omega)$, for $L - a.e.t.$ Thus, both V and F are linear in the in the relevant range. In other words, if there are intertemporal sunspot equilibria, the representative agent economy is without essential loss of generality, semi-linear.

6 Non Existence of Sunspot Equilibria

The converse of proposition 2 (point 1) is not true. There are intertemporal competitive equilibria of economies (U, F, x_0) that do not allow for an equivalent CPSE. Equivalently, by proposition 2 (point 2), there are intertemporal competitive equilibrium allocations of (U, F, x_0) that cannot be supported as intertemporal sunspot equilibria. In this section, we construct examples by using the static notion of equilibrium. However, before doing that we need to clarify the relation between static and intertemporal sunspot equilibria.

6.1 Static and Stationary Sunspot Equilibria

As already mentioned, efficient static allocations are efficient, stationary intertemporal allocations. Also, static competitive equilibria are stationary, intertemporal competitive equilibria. However, there might be static sunspot equilibria that are not intertemporal (stationary) sunspot equilibria. The reason is simple. In the static formulation of the model time is absent, while in the intertemporal version, agents can use time to generate lotteries over basic commodities. When prices are, as in the sunspot case, linear, the agents can reproduce lotteries that yield the least concave utility function. Hence, only static sunspot equilibrium allocations yielding the least concave utility can survive the test of time, i.e., they are stationary, intertemporal sunspot equilibria. This is summarized in the next proposition.

Proposition 4. *A static sunspot equilibrium of the economy* $(U, F, \hat{B})$ *is a stationary, intertemporal sunspot equilibrium only if it is a static sunspot equilibrium of the "concavified" economy* $(V, F, r; \hat{B})$ *(as well as, a stationary, intertemporal sunspot equilibrium of the concavified economy* $(V, F, r; \hat{B})$*).*

Proof. The argument is a trivial consequence of proposition 2.

6.2 Examples

We construct (open sets of) static economies (U, F) that have:

1) a continuum of Pareto ranked static sunspot equilibria associated to distinct specifications of the sunspot space;

2) efficient allocations supportable by static sunspot equilibria, but not by CPSE, and an empty set of intertemporal sunspot equilibria for some value of the initial capital stock x_0 and for all $\hat{B} \subset B$;

3) an empty set of sunspot static equilibria.

By proposition 2, the Pareto ranked static sunspot allocation cannot be intertemporal sunspot allocations and intertemporal sunspot equilibria are, without loss of generality, CPSE. Thus, point 1) does not apply to the intertemporal economy, while 2) implies that sunspot and lottery intertemporal equilibrium allocations are not equivalent and that the set of intertemporal sunspot equilibria may be empty. Thus, 1), 2) and 3) point out that, when preferences are not convex, static sunspot equilibria are a inadequate tool to analyze stationary equilibrium allocations of growth economies.

Proposition 5. *Static sunspot equilibria may fail to exist. When they exist they may be indeterminate and inefficient. Efficient stationary allocations may be decentralized as static sunspot equilibria, but not as static CPSE.*

Example 3. An economy may have a continuum of static sunspot equilibria, all Pareto ranked.

The static economy is described by $C = K = 1$, and $F(x, y) = f(x) - y^{\theta}$, $\theta \geq 1$, with f strictly concave and such that $f'(1) = r$ and $f(1) = 1$. The utility function has $U'(0) = +\infty$, is monotonically increasing in an interval $[0, M]$, and is such that $\text{co}\, U(1) > U(1)$. For example the following utility function satisfies all these conditions:

$$U(y) = \min\{\max\{y^{\alpha}, y\}, 2M - y\},$$
$$\text{for } \alpha \in (0, 1) \text{ and } M > 2 \text{ and arbitrarily large} \tag{14}$$

We restrict the space of extrinsic uncertainty to two points $\omega \in \{1, 2\}$ with $\sigma(1) = \sigma$. We set $q = 1/\theta$. The first order conditions for the firm's problem (recall that this problem is concave) give that the equilibrium capital and the firm supply are

$$\hat{x} = 1,\ \hat{y}_f(\omega) = (\hat{p}(\omega))^{\frac{1}{\theta-1}},\ \omega = 1, 2, \tag{15}$$

when $\theta > 1$. When $\theta = 1$, the equilibrium prices are $p(\omega) = 1$, $\omega = 1, 2$. At the equilibrium prices, the set of optimal y's is the set of non negative quantities.

For arbitrarily given $\theta \geq 1$, an equilibrium is a sunspot allocation $\hat{y}^* = (\hat{y}^*(1), \hat{y}^*(2))$ such that:

1. the solution $\hat{y}$ of the consumer's problem

$$\max_{\hat{y}} \sum_{\omega} \sigma(\omega) U(\hat{y}(\omega)),\ \text{subject to}$$

$$\sum_{\omega} \sigma(\omega)(\hat{y}^*(\omega))^{\theta-1}\hat{y}(\omega) \leq \sum_{\omega} \sigma(\omega)(\hat{y}^*(\omega))^{\theta}$$

   coincides with $\hat{y}^*$,
2. the market clear: $\sum_{\omega} \sigma(\omega)(\hat{y}^*(\omega))^{\theta} = 1$.

Consider for example the utility function in equation (14). Note that the equilibria are symmetric: to any equilibrium at $(\sigma, 1 - \sigma))$ corresponds an equilibrium at $(1 - \sigma, \sigma)$, so we only consider $\sigma \equiv \sigma(1) \leq 1/2$. The Inada condition that $U'(0) = +\infty$ insures that $\hat{y}^* \gg 0$. Let, without loss of generality, $\alpha p_2 < p_1$. (Otherwise relabel the states) Since, $\lim_{y\uparrow 1} U'(y)/p(1) = \alpha/p(1) < \lim_{y\downarrow 1} U'(y)/p(2) = 1/p(2)$, the budget feasible allocation $y(\omega) = 1$, $\omega = 1, 2$ is not optimal. When $\max\{\frac{1}{\sigma\hat{p}(1)}, \frac{1}{(1-\sigma)\hat{p}(2)}\} < M$, the utility function is monotonically increasing in the budget feasible set, so the solution of the consumer problem will satisfy the budget constraint as an equality, and so we only have to find $\hat{y}^*$ that coincide with the solution of

$$\max_{(y_1, y_2)} \sum_{\omega} \sigma(\omega) U(\hat{y}(\omega)),\ \text{subject to}$$

$$\sum_{\omega} \sigma(\omega)\hat{p}^*(\omega)\hat{y}(\omega) = 1,\ \text{for}\ \hat{p}^*(\omega) = (\hat{y}^*(\omega))^{\theta-1}.$$

The first order condition that marginal utilities are *equal* at the two choices y_j is necessary: note that the probabilities appear both in the utility and in the budget constraint, and therefore cancel out. The sunspot equilibrium allocation $\hat{y}^*$ and the supporting prices $\hat{p}^*$ are a solution to the following system of equations:

$$\hat{y}(1) = (\frac{\alpha\hat{p}(2)}{\hat{p}(1)})^{1-\alpha},\ \sum_{\omega} \sigma(\omega)\hat{p}(\omega)\hat{y}(\omega) = 1,\ \text{and}\ \hat{p}(\omega) = (\hat{y}(\omega))^{\theta-1},\ \omega = 1, 2. \tag{16}$$

The latter is a system of 4 equations in 5 unknowns $(\hat{y}, \hat{p}, \sigma)$. The solution changes with σ and this gives the continuum of equilibria. The value depends on σ, and the equilibria are Pareto ranked.

The case $\theta = 1$ is simple:

1. there is an equilibrium for any $\sigma \in (1/M, 1/2]$;
2. the allocations are different for different σ:
$$\hat{y} = (\frac{1-(1-\sigma)\alpha^{\frac{1}{1-\alpha}}}{\sigma}, \alpha^{\frac{1}{1-\alpha}}); \tag{17}$$
3. the value is strictly decreasing in σ, and given by:
$$1 + (1-\sigma)(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}). \tag{18}$$

The allocations described in 2. satisfies $\hat{y}(2) \in (0,1)$ and $y(1) \in (1, M)$, for $\sigma \in (1/M, 1/2]$. The matrix of partial derivatives of the system of equations 16 with respect to $(\hat{y}, \hat{p})$ is for, $\theta = 1$ and for any $\sigma' \in (1/M, 1/2]$, invertible. Hence, for $\sigma' \in (1/M, 1/2]$, by the implicit function theorem, for each (θ, σ) in a neighborhood of $\theta = 1$ and σ', there exists a sunspot equilibrium. ∎

The efficient lottery of the static economy, in the next example, cannot be supported by linear prices and, hence, by a CPSE. Quite surprising, it can be supported by sunspot equilibria that display sunspot price variability. As already explained, this implies that the set of sunspot static equilibria is empty. This example is in contrast with the findings of [5] for finite, pure exchange economies. The existence of sunspot equilibria which are (genuinely) not CPSE is, in that context, possible only with finite sunspot state space and discrete consumption sets.

Example 4. An efficient allocation of a static economy may be supported as a sunspot equilibrium, but not by a CPSE.

We consider the economy defined by technology of the previous example and the utility function in equation (14), with parameter values (α, θ), $\theta > 1$ and $\alpha \in (0,1)$. As already explained, the optimal amount of capital is $x^* = 1$, for all θ. The optimal lottery β^* is found by solving the following steps:

1) for given $q > 0$, find the set $Y^*(q) = \arg\max_y U(y) - qy^\theta$

2) find a price for the investment good q and a probability distribution over $Y^*(q)$ such that $y^\theta(\beta) = 1$ for $\beta \in M_{1,+}(Y^*(q))$. Recall that, by definition, for y and $y' \in Y^*(q)$, $U(y') - q(y')^\theta = U(y) - q(y)^\theta$.

Since $\lim_{y\uparrow 1} U'(y) = \alpha < \lim_{y\downarrow 1} U'(y) = 1$, the deterministic allocation $\beta^* = \delta_1$ is suboptimal for all parameter values, $\alpha \in (0,1)$ and $\theta > 1$. The next claim shows that, given the initial condition $x_0 = 1$, the efficient allocation is a unique, stationary and non degenerate lottery. The proof of the Claim is in the appendix.

Claim. The intertemporal economy $(U, f(x) - y^\theta, 1)$ has a unique efficient allocation $(\hat{x}, \hat{a}, \hat{\beta})$, $\hat{x}(t) = 1$, $\hat{a}(t) = 0$ and $\hat{\beta}(t) = \beta^*$, for $L - a.e.t.$ The support of the optimal lottery β^* is $\{y_1^*, y_2^*\}$, with $0 < y_1^* < 1 < y_2^* \leq M$.

In order to avoid unnecessary complications, we restrict the parameters values in the set $P(M) = \{(\alpha, \theta) : \alpha \in (0,1),\ \theta > 1 \text{ and } y_2^* < M\}$, an open subset of $\Re_+^2$. To characterize the optimal lottery we seek values of the four unknowns (y_1, y_2, β, q) that satisfy four equations (market clearing, two optimality conditions, $y_i \in Y(q)$, and $U(y_1) - q(y_1)^\theta = U(y_2) - q(y_2)^\theta)$. By straightforward computations, the unique solution is:

$$y_1^* = (\tfrac{\theta-\alpha}{\theta-1})^{\frac{\theta-1}{\theta(1-\alpha)}}(\alpha)^{\frac{1}{\theta(1-\alpha)}},\ y_2^* = (\tfrac{\theta-\alpha}{\theta-1})^{\frac{\theta-\alpha}{\theta(1-\alpha)}}(\alpha)^{\frac{\alpha}{\theta(1-\alpha)}},$$
$$q^* = \theta^{-1}(\tfrac{\theta-1}{\theta-\alpha})^{\frac{(\theta-\alpha)(\theta-1)}{\theta(1-\alpha)}}(\alpha)^{\frac{-\alpha(\theta-1)}{\theta(1-\alpha)}} \text{ and } \beta^* = \frac{y_2^{*\theta}-1}{y_2^{*\theta}-y_1^{*\theta}}$$

For given $\theta > 1$, both y_1^* and y_2^* are strictly increasing functions of α. For $\alpha = 0$, $y_2^* = \frac{\theta}{\theta-1} > 1$ and $y_1^* = 0$, while for $\alpha = 1$, $y_1^* = 1$ (and $y_2^* = \infty$).

In order to show that the efficient stationary allocation can be supported as a static sunspot equilibrium, we take $\Omega = \{1,2\}$ with $\sigma = \beta^*$. Profit maximization pins down the supporting prices $\hat{p}$ as:

$$\hat{p}^*(\omega) = q^*\theta(y_\omega^*)^{\theta-1},\ \omega = 1,2.$$

Since both y_1^* and y_2^* belong to regions where the utility function is concave, $\frac{p^*(1)}{p^*(2)}$ is equal to the value of the marginal rate of substitution computed at the sunspot efficient allocation. Furthermore, for each given $M > 2$, $\frac{p^*(1)}{p^*(2)} \neq 1$, for an open and dense set of parameter values in $P(M)$. Hence, for (α, θ) in an open and dense subset of $P(M)$, efficient allocation are supported as sunspot equilibria, but not as CPSE. Thus, by proposition 2, the uniqueness of the efficient allocation implies that, for these parameter values, the set of sunspot equilibria is empty. ■

In Proposition 2, we were silent about the possibility of the sunspot equilibria of the static economy to support the efficient allocations. The last example shows that indeed CPSE may fail to accomplish this task. However, it leaves open this possibility for different types of sunspots equilibria. Unfortunately, there is a fundamental reason that breaks the equivalence result: sunspot equilibria may not exists. This is shown in the next example.

Example 5. An economy (U, F, r) may have neither sunspot equilibria nor intertemporal sunspot equilibria for some initial condition x_0 and any $\sigma-$algebra $\hat{B} \subset B$.

Let $C = K = 2$, and

$$F_1(x_1, y_1) = \frac{1}{2}(f_1(x_1) - y_1^2) \tag{19}$$

$$F_2(x_2, y_2) = f_2(x_2) - y_2 \tag{20}$$

while the preferences are defined by

$$U(y_1, y_2) = y_2 + u(y_1), \tag{21}$$

where for $\alpha < 1$ and $\epsilon > 0$ and small enough, u is defined by

$$u(y_1) = \begin{cases} (\frac{11}{1-\epsilon})y_1 - (10+\epsilon), & for\ y_1 \in [0, 1-\epsilon), \\ y_1 + (\alpha/2)[(y_1-1)^2 - (\epsilon)^2], & for\ y_1 \in [1-\epsilon, 1+\epsilon] \\ y_1, & for\ y_1 > 1+\epsilon. \end{cases}$$

The function u is continuous and piecewise differentiable, and therefore so is U; for ϵ small enough, U is strictly increasing.

Since the functions F_i are, for $i = 1, 2$, separable in x and y, the optimal capital stocks are independent of prices and are found by solving:

$$\max_{x_i} a_i f_i(x_i) - r x_i,$$

for $i = 1, 2$ and $a_1 = 1/2$ and $a_2 = 1$. Since f_i are strictly concave the optimizers x_i^* are unique and we take them to satisfy:

$$f_i(x_i^*) = x_i^* = 1, \text{ for } i = 1, 2.$$

In this static economy, although the utility function is not concave, the planner allocation, (x^*, y^*), is deterministic, it is equal to $(1, 1, 1, 1)$ and it is supported by prices $(q_1^*, q_2^*) = (1, 1)$. Since the characterization of the efficient allocation plays a minor role in the proof of the non-existence, we postpone its detailed construction to the appendix, and we just point out the only property needed in the sequel, namely:

$$1 \in \arg\max u(y_1) - \frac{1}{2} y_1^2$$

In the static economy, we take $([0, 1), B, L)$ as the probability space of sunspots. We normalize prices by setting $q_1 = 1$.

Let $(x, \hat{y}_f)(p, q)$ and $\hat{y}_h(q, p)$ denote, respectively, the firm's profit maximizing and the household's utility maximizing choices, for given price function $\hat{p} : [0, 1) \to \Re_{++}$ and $q \in \Re^K_{++}$. Since the functions F_i are, for $i = 1, 2$, separable in x and y, the profit maximizing capital stocks are :

$$f(x_i^*) = x_i^* = 1, \text{ for } i = 1, 2.$$

Given the adopted specification of F, $(x, \hat{y}_f) \in (x, \hat{y}_f)(p, q)$ if

$$\hat{y}_{1,f}(s) = \hat{p}_1(\omega),\ \hat{p}_2(\omega) \leq q_2 \text{ and } \hat{p}_2(\omega)\hat{y}_{2,f}(\omega) = q_2 \hat{y}_{2,f}(\omega),\ L - a.e.\omega \quad (22)$$

Let $\hat{y}_h(p, q)$ denote the optimal consumption allocation chosen by the household at prices (p, q).

The market clearing conditions are:

$$\begin{aligned} &(\int_{[0,1)} ((\hat{y}_{1f}(\omega))^2, \hat{y}_{2f}(\omega)) d\omega = (f(\hat{x}_1), f(\hat{x}_2)) = (1, 1) \\ &\text{and } \hat{y}_f(\omega) = \hat{y}_h(\omega),\ L - a.e.\omega \end{aligned} \quad (23)$$

Now we give immediately the final step in the proof of the non-existence of equilibrium, assuming first the following two preliminary steps. Later (see equation (27)) we prove that at a sunspot equilibrium:

$$\int_{[0,1]} \hat{p}_1(\omega)d\omega \geq \int_{[0,1]} \hat{p}_1(\omega)\hat{y}_1(\omega)d\omega. \tag{24}$$

We also prove (in lemma (2)) that any allocation $\hat{y}$, with $\hat{y}_1(\omega) = 1$, for $L - a.e.\omega$, cannot be an equilibrium allocation. We assume for the moment these two preliminary results, and show the conclusion.

Proposition 6. *There are no sunspot equilibria of the static economy defined by (19), (20), (21).*

Proof. By contradiction, let $\hat{y}$, $\hat{y} = \hat{y}_f = \hat{y}_h$, be the equilibrium allocation. Since by the firm first order conditions $\hat{p}_1(\omega) = \hat{y}_1(\omega)$, $L - a.e.\omega$, (24) can be rewritten as:

$$\int_{[0,1)} \hat{p}_1(\omega)d\omega = \int_{[0,1)} \hat{y}_1(\omega)d\omega \geq \int_{[0,1)} (\hat{y}_1(\omega))^2 d\omega \tag{25}$$

Hence, by the market clearing conditions (23) :

$$\int_{[0,1)} \hat{y}_1(\omega)d\omega \geq \int_{[0,1)} (\hat{y}_1(\omega))^2 d\omega = 1. \tag{26}$$

By Jensen's inequality, the last inequality holds true if and only if $\hat{y}_1(\omega) = 1$, $L - a.e.\omega$. However, lemma (2) below rules out that this latter can be a sunspot equilibrium allocation.

To proceed in our argument, we show that any consumption allocation $\hat{y}$, with $\hat{y}_1(\omega) = 1$, $L - a.e.\omega$, cannot be a sunspot equilibrium allocation. The argument is based on a simple observation: at the consumption bundle $(1, y_2)$, $y_2 \geq 0$, the utility function is not concave and, hence, no linear price can support that allocation.

Now the details. Arguing by contradiction, suppose that $(x, \hat{y})$ is an equilibrium allocation. Let $(q, \hat{p})$ be the equilibrium price function. Since $x = (1, 1)$ and $\int_{[0,1]} F(x, \hat{y}(\omega))d\omega = 0$, the value of the firm net of capital expenses is:

$$v(\hat{p}, q) = \int_{[0,1)} p(\omega)\hat{y}_f(\omega)d\omega = 1 + q_2.$$

Let $\hat{y}^*$ be any consumption allocation with $\hat{y}_1^*(\omega) = 1$, $L - a.e.\omega$. The proof of the next Lemma is deferred to the appendix.

Lemma 2. $\hat{y}^*$ *can not be a sunspot equilibrium allocation of the static economy.*

To conclude the construction we have to show that if there were a sunspot equilibrium then inequality (24) would hold true. Evidently, by proposition 6, the latter is a contradiction.

Suppose that there exists a sunspot equilibrium. Recall that $(1,1)$ is the efficient consumption bundle. Then, by 2 and by the definition of efficient allocation, $U(1,1) \geq \int_{[0,1]} U(\hat{y}(\omega))d\omega$. Hence, by revealed preferences, it must be that the constant allocation $\hat{y}^*$, with $\hat{y}^*(\omega) = (1,1)$, $L - a.e.\omega$, is at least as expensive as the equilibrium allocation $\hat{y}$. Hence, at the equilibrium prices $\hat{p}$, it must be:

$$\int (\hat{p}_1(\omega) + \hat{p}_2(\omega))d\omega \geq \int (\hat{p}_1(\omega)\hat{y}_1(\omega) + \hat{p}_2(\omega)\hat{y}_2(\omega))d\omega \qquad (27)$$

However, by taking into account that $\int \hat{p}_2(\omega)\hat{y}_2(\omega)d\omega = \int \hat{p}_2(\omega)d\omega$ (see 23), the latter inequality simplifies to (24).

To conclude the analysis we show that given the initial condition $x_0 = (1,1)$, there cannot exist sunspot equilibria of the intertemporal economy. Suppose, by contradiction, that there exists an intertemporal sunspot equilibrium. By propositions 2, the intertemporal sunspot equilibrium is efficient and , without loss of generality, a CPSE. Furthermore, there exists an equivalent lottery equilibrium of the intertemporal economy (U, F, r). Hence, for given initial condition $x(0) = x_0$, the economy (U, F, r) has two efficient paths, one stationary $(\hat{x}, \hat{\beta})$, with $(1, 1, \delta_{(1,1)}) = (\hat{x}, \hat{\beta})(t)$, for all t, and one not, $(\hat{x}^*, \hat{\beta}^*)$. Consider for $\lambda \in (0,1)$, the path $(\lambda\hat{x} + (1-\lambda)\hat{x}^*, \lambda\hat{\beta} + (1-\lambda)\hat{\beta}^*)$, where $(\lambda\hat{\beta} + (1-\lambda)\hat{\beta}^*)(t)$ is the compounded lottery that assigns with probability λ the lottery $\hat{\beta}(t)$ and $(1-\lambda)$ the lottery $\hat{\beta}^*(t)$. Evidently, by efficiency and linearity in lotteries, $\int U(\hat{\beta}(t))dt = \int U(\hat{\beta}^*(t)dt = \int U(\lambda\hat{\beta} + (1-\lambda)\hat{\beta}^*)(t)dt$. However, by the concavity of F,

$$F((\lambda\hat{x}+(1-\lambda)\hat{x}^*, \lambda\hat{\beta}+(1-\lambda)\hat{\beta}^*)(t)) \geq \lambda F((x,\hat{\beta})(t))+(1-\lambda)F((x^*,\hat{\beta}^*)(t)) = \lambda\dot{x}(t) + (1-\lambda)\dot{x}^*(t) = (1-\lambda)\dot{x}^*(t).$$

If the inequality is strict for an $L-$ positive measure set of $\Re_+$, the strict monotonicity of U contradicts the efficiency of both paths. Hence, for $L-a.e.t$, the previous inequality must be an equality. Then, the strict concavity of F in x and of F_1 in y_1, implies that $\hat{x}(t) = \hat{x}^*(t)$ and $\hat{\beta}^*(t)\{y : y_1 = 1\} = 1$, for $L - a.e.t$.

But then given the form of F, $y(\hat{\beta}^*(t)) = 1$, for $L - a.e.t.$, and since U (and F) is linear in y_2, $\hat{\beta}^*(t)$ and $\hat{\beta} = \delta_{1,1}$ yield the same utility and $F(1, 1, \hat{\beta}^*(t)) = F(1,1,1,1)$, for $L-a.e.t$. Hence, the path $(\hat{x}^*, \hat{\beta}^*)$ is supported by the supporting prices of the stationary allocation $(\hat{x}, \hat{\beta})$. Since the latter can not be supported by linear prices, the intertemporal economy does not have sunspot equilibria. ∎

7 The Large Economy

In this section we study the version of the economy with a large number of identical households. For large numbers of individuals we mean (as usual) that the law of large numbers holds true. Namely, if we assign to each household an *i.i.d.* lottery $\beta \in M_{1,+}(Y)$, the pro-capita average allocation is equal (with probability one) to $y(\beta) = \int y\beta(dy)$. The particular specification the measurable space of households is not important, provided that the law of large numbers holds. We take it to be $\{[0,1), B, L\}$ (with the usual caveat that applies to the validity of the Law of Large numbers with a continuum of random variables). As for any large economy, we assume that feasibility is a restriction over the random realizations of the pro-capita total output.

Since we are dealing with a market economy where identical individual face identical choices, we limit attention to efficient allocations yielding the same level of utility to each household, "equal utility allocations." The set of consumption allocations, at each instant, can be identified just with the set $M_{1,+}(Y)$. To each $\beta \in M_{1,+}(Y)$ it is associated the pro-capita consumption, $y(\beta)$. Hence, feasibility is just a restriction over average output. The reason is obvious. If we assign to each household independent lotteries identically equal to β, by the law of large number, the average output is equal (with probability 1) to $y(\beta)$. Considering more general random allocations does not enhance efficiency, [14].

A large economy has basically the same properties of a semi-linear economy. Since feasibility restricts the average values of the lotteries, the basic properties of equilibrium and efficient path of the economy (U, F, r) are closely captured by those of the concavified economy (V, F, r). This is the substance of the arguments, which, are obviously very similar to the ones used for the semi-linear case.

7.1 Equal Utility Efficient Paths

Equal utility efficient paths are the solution to the following programming problem:

$$\max_{(\hat{x},\hat{\beta})} \int e^{-rt} U(\hat{\beta}(t))dt \text{ subject to } F(\hat{x}(t), y(\hat{\beta}(t))) \geq \dot{x}(t), \hat{x}(0) = x_0. \quad (28)$$

The optimal solutions to (28) can be characterized in a simple way by making reference to the "concavified" economy (V, F, r) Hence, consider the following programming problem, where, given the concavity of V, the planner optimizes within deterministic allocations:

$$\max_{(\hat{x},\hat{y})} \int e^{-rt} V(\hat{y}(t))dt \text{ subject to } F(\hat{x}, \hat{y}(t)) \geq \dot{x}(t), \hat{x}(0) = x_0. \quad (29)$$

Let $V(x_0)$ be the value of (29) and $U(x_0)$ be the value to (28). Then:

Proposition 7. *$(\hat{x}, \hat{y})$ is an optimal solution to (28) if and only if $(\hat{x}, \hat{\beta})$ with $U(\hat{\beta}(t)) = V(\hat{y}(t))$ and $y(\hat{\beta}(t)) = \hat{y}(t)$, for all $L-a.e.t$, is an optimal solution to (29). Hence, $V(x_0) = U(x_0)$.*

Proof. The proof is an easy consequence of the definition of least concave function.

7.2 Lottery and Sunspot Equilibria

The large competitive economy with a complete set of markets for lotteries is identical to the one we have already described. The only difference is in the presence of a large number of households. In order to characterize the equilibria of our large economy, we follow the same technique adopted for the characterization of efficient paths. We study the "concavified" economy where, rather than U, V is the utility function of the households.

The efficient allocations of the concavified economy and the efficient allocations of the large economy are equivalent in the sense of proposition 7. Furthermore, for both economies, competitive allocations and efficient allocations coincide. Hence, the two economies have equivalent competitive allocations. In the next proposition, we state that they have identical supporting prices. Therefore, supporting prices of large economies are linear in commodities. This is the key fact for sunspot decentralization of the efficient paths.

Proposition 8. *Let $(\hat{x}, \hat{a}, \hat{y})$ be an efficient path of the "concavified" economy supported by prices $(\hat{q}, \hat{b}, \hat{p})$. Then the efficient path $(\hat{x}, \hat{a}, \hat{\beta})$, with $U(\hat{\beta}(t)) = V(\hat{y})$ and $y(\hat{\beta}(t)) = \hat{y}(t)$, of the large economy is supported by the same prices.*

Proof. The proof is basically identical to the proof of proposition 3 and it is, therefore, omitted. ∎

Constant Prices Sunspot equilibria

Propositions 3, 7 and 8 establish a strong similitude between representative economies with semi-linear technologies and economies with a large number of households. It is not a surprise, therefore, that the results of Corollary 1 generalize to large economies. However, there is a difference between this two environments that makes the extension non immediate. In the representative household semi-linear environment, aggregate and individual consumption are obviously equal. Consumption may be random, although only its average value is of importance for feasibility. In the large economy, individual and pro-capita consumption are very different. The first may display randomness, while the second is deterministic and equal to the average value (over lottery realizations) of the random individual consumptions. Hence, sunspot allocation have in the large economy the additional task of creating enough correlation among individual allocations in order to make the pro-capita consumption sunspot independent and feasible. A sunspot consumption allocation of the large economy is a measurable map $\hat{A} : [0,1)^2 \times R_+ \to Y$, where $\hat{A}(h,\omega,t)$ denotes the consumption bundle consumed by individual h at t when ω realizes. Given an initial condition x_0, *an* intertemporal $CPSE$ of the large economy is an $\omega - invariant$ trajectory of prices $(\hat{q},\hat{b},\hat{p})$, of investments, capital goods and pro-capita consumption allocations $(\hat{x},\hat{a},\hat{y})$ and a consumption allocation $\hat{A}$ such that *i)* $(\hat{x},\hat{a},\hat{y})$ is profit maximizing at $(\hat{q},\hat{b},\hat{p})$; *ii)* $(\hat{x},\hat{a},\hat{A}(h))$ is utility maximizing at $(\hat{q},\hat{b},\hat{p})$, for $L - a.e.h$; *iii)* $\int_{[0,1)} \hat{A}(h,\omega,t)dh = \hat{y}(t)$, $L \times L - a.e.(\omega,t)$. *iv)* $F(\hat{x}(t),\hat{y}(t)) \geq \hat{a}(t)$, $L - a.e.t$. We exploit propositions 7 and 8. Hence, for given initial condition, we take a competitive equilibrium allocation of the concavified economy and we show that the same prices support a $CPSE$ with equivalent allocation. This result implies, by propositions 7 and 8, that any efficient (lottery) allocation of the large economy can be equivalently supported by a $CPSE$.

The following observation, although not necessary (see for instance [5]), greatly simplifies the argument. Consider the set $\Xi = \{(y,u) : y \in Y$ and $u = U(y)\} \subset R^{C+1}$. Let $CH(\Xi) \subset R^{C+1}$ be the convex hull of Ξ. Consider an equilibrium allocation of the "concavified" economy. For each t, by the definition of least concavity, $(\hat{y}(t),V(\hat{y}(t))) \in CH(\Xi)$. Then, Caratheodory's theorem implies that there exists at most $C+2$ consumption bundles $(\hat{y}_1(t),....,\hat{y}_{C+2}(t))$ and $\hat{\beta}(t) \in \Delta(C{+}2)$ such that $\Sigma_\kappa \hat{\beta}_\kappa(t)\hat{y}_\kappa(t) = \hat{y}(t)$ and $\Sigma_\kappa \hat{\beta}_\kappa(t)U(\hat{y}_\kappa(t)) = V(\hat{y}(t))$. Hence, when we analyze efficient (lottery) allocations of the large economy there is no loss of generality in restricting attention to allocations $\hat{\beta}(t)$ with support consisting of at most $C+2$ points.

Proposition 9. *Let $(\hat{q},\hat{b},\hat{p};\hat{x},\hat{a},\hat{y})$ be an intertemporal equilibrium of the concavified economy (V,F,x_0). Then, there is a sunspot allocation $\hat{A}$, with*

$$\int \hat{A}(h,\omega,t)d\omega = \hat{y}(t),$$

and $\int_{[0,1)} U(\hat{A}(h,\omega,t))d\omega = V(\hat{y}(t))$, for $L \times L - a.e.(h,t)$, such that the vector $(\hat{q},\hat{b},\hat{p};\hat{x},\hat{a},\hat{y},\hat{A})$ is an intertemporal CPSE of the large economy.

Proof. By proposition 3 and corollary 1, at $(\hat{q},\hat{b},\hat{p})$ households maximize utility by selecting the capital and investment trajectories $(\hat{x},\hat{a})$ and an individual sunspot allocation $\hat{y}^*$ such that

$$\int_{[0,1)} U(\hat{y}^*(t,\omega))d\omega = V(\hat{y}(t))$$

and $\int_{[0,1)} \hat{y}^*(t,\omega)d\omega = \hat{y}(t)$. Hence, we just have to show, that there exists a function $\hat{A}(.)$ such that $\hat{A}(h) = \hat{y}^*$, for all h, and $\int_{[0,1)} A(h,\omega,t)dh = \hat{y}(t)$, $L \times L - a.e.(t,\omega)$. By Caratheodory's theorem, for each $\hat{y}(t)$ there exist $\hat{\beta}(t) \in$

$\Delta(C+2)$ such that $U(y(\hat{\beta}(t))) = V(\hat{y}(t))$ and $y(\hat{\beta}(t)) = \hat{y}(t)$. Denote by $\hat{y}_\kappa(t)$, $\kappa = 1, ..., C+2$, the points in the support of $\hat{\beta}(t)$. Divide, for each t, the set of sunspots $[0,1)$ in the $C+2$ disjoint intervals, $I_\kappa(t) = [\Sigma_{j=0}^{\kappa-1}\hat{\beta}_j(t), \Sigma_{j=0}^{\kappa}\hat{\beta}_j(t))$, with the convention that $\hat{\beta}_0(t) = 0$. Let $\hat{+}$ denote addition modulo 1. Then $\hat{A}$ is defined as:

$$\hat{A}(h, \omega, t) = \hat{y}_\kappa(t),\ if\ \omega \hat{+} h \in I_\kappa(t).$$

Evidently, $\hat{A}(h) = \hat{y}^*$, for all h, and $\int_{[0,1)} \hat{A}(h, \omega, t)dh = \hat{y}(t)$, since $L\{h : A(h, \omega, t) = y_\kappa(t)\} = L\{h : h\hat{+}\omega \in I_\kappa(t)\} = \hat{\beta}_\kappa(t)$, for each pair (κ, t). ∎

8 Indeterminacy of Sunspot Equilibria

We construct an example of a large static economy with a countable infinity of sunspot equilibrium allocations, which are Pareto ranked. Once again, inefficient sunspot equilibria of the large static economy can never be stationary sunspot equilibria of the large economy. However, there is a significant difference between example 3 and this example, which justifies its construction. The static economy with a representative household bears no similarity with any finite dimensional Arrow-Debreu economy. This is particularly evident when equilibrium calls for random consumption allocations. The latter maps into randomness of the investment allocations, $a = F(x, y)$. However, feasibility requires that investment be equal to zero in average. It is the feasibility requirement that makes the representative household economy very different than anything else. Of course, its interest is justified by its relation with the associated dynamic economy. The situation is quite different for a large economy. Randomness in individual consumption allocations does not translate (at efficient allocations) into randomness of investment allocations. When pro-capita output is deterministic, feasibility in the static large economy is identical to feasibility in any finite dimensional Arrow-Debreu economy with a large number of identical households. In particular, it is immediate to show that for specific choices of the map F, the equilibrium of the large static economy is an equilibrium of a large pure exchange economy (with identical individuals) with the same preferences and endowments equal to the efficient pro-capita aggregate consumption of the static economy. We make sure that this feature is present in our example. Hence, the Pareto ranked multiple equilibria of the large static economy are equilibria of the "isomorphic" pure exchange economy.

8.1 The Economy

There is a unique consumption good and a unique capital good. The technology is $F(x, y) = 1/2(f(x) - y^2)$. f is chosen so that for $x^* = \arg\max(f(x)/2) - rx$, $f(x^*) = 1$. We identify the space of sunspots with a set of J points having identical probability $1/J$ for some integer $J > 1$. The prices of the consumption good are restricted to be sunspot invariant and normalized to one. The form of the technology F and the restrictions on sunspots and prices immediately imply that $q = 1$ and the output of the firm is equal to 1. Hence, the economy is isomorphic to a standard pure exchange economy populated by identical households with one unit of endowment of the only existing good. There is a unique consumption good and a continuum of identical households, with generic index h in the space $\{[0,1), B, L\}$. The utility function U is piecewise linear and non decreasing[5], defined as:

$$U = \begin{pmatrix} 0, \text{ for } y \in [0, y_*], \\ \alpha(y - y_*), \text{ for } y \in [y_*, y_b], \\ \alpha(y_b - y_*) + \gamma(y - y_b), \text{ for } y > y_b. \end{pmatrix}$$

We assume:

$H1)\ y_* < 1 < y_b,\ H2)\ 1 > \delta = \alpha(y_b - y_*)/y_b > \gamma$ and $H'2)\ \alpha(1 - y_*) - \gamma > 0.$

$H1$ allows for the existence of sunspot equilibria. By $H2$ there is a least concave function V associated to U. $H2$ and $H'2$ are not essential, but they simplify the computations. The least concave function associated with U is:

$$V = \begin{pmatrix} \delta y, \text{ for } y \in [0, y_b], \\ \alpha(y_b - y_*) + \gamma(y - y_b), \text{ for } y > y_b. \end{pmatrix}$$

There is a complete set of contingent commodity markets, where individuals buy contingent commodity bundle $y^J = (y_1, ..., y_J)$. In the next two sections we drop the subscript h. Since states are equiprobable and prices sunspot invariant and equal to 1, the programming problem of the households is:

$$(H^J) \qquad \max_{y^J} \Sigma_j U(y^j), \text{ subject to } \sum_j (y_j - 1) = 0.$$

Let $y_h(J)$ be the set of optimal solutions to (H^J). An allocation A^J is a measurable map from $[0,1)$ to R_+^J. $A^J(h)$, $h \in [0,1)$, is the vector of contingent commodities of individual h prescribed by the allocation map A^J.

An allocation A^J is feasible if

$$\int_{[0,1]} A^J(h,j) L(dh) = 1, \text{ for all } j = 1, ..., J.$$

Given J, a sunspot equilibrium is a feasible allocation such that $A^J(h) \in y(J)$, for $L - a.e.h.$

[5] It should be evident from the analysis that, at the cost of heavier computations, all the conclusions can be generalized to strictly increasing and non concave functions.

8.2 Individual Optimization Problem

Facing several equiprobable states and sunspot invariant prices, households optimize by selecting few consumption bundles and making endogenous their probabilities by associating an endogenous number of states to each distinct consumption bundle. This idea is formalized by the next lemma the proof of which is in the appendix. Let $S(y^J) = \{y \in R_+ : y_j = y, \text{ for some } j\}$, $y^J \in y(J)$.

Lemma 3. *$\#S(y^J) \leq 2$, for all J and some $y^J \in y(J)$.*

Because of Lemma 3, in order to get insights into the optimal solution to (H^J), we can study, first, a fictitious and simple programming problem with two states of uncertainty and with $\sigma \in (0, 1)$ denoting the probability of the first state. The two prices are sunspot invariant and equal to 1. The household solves:

$$(H^\sigma) \ \max \sigma U(y_1) + (1-\sigma)U(y_2), \ \textit{subject to } \sigma y_1 + (1-\sigma)y_2 \leq 1$$

The set of optimal solution to (H^σ) is denoted by $\hat{y}(\sigma) = (y_1, y_2)(\sigma)$. Evidently, $(y_1, y_2) \in \hat{y}(\sigma)$ if and only if $(y_1, y_2) \in \hat{y}(1-\sigma)$. Hence, we just analyze the problem for $\sigma \in [1/2, 1)$.

Let σ^* be such that

$$(1-\sigma^*)\hat{y}_b = 1$$

By $H1$, $\sigma^* \in (0, 1)$ and since $(1-\sigma^*)U(y_b) = V((1-\sigma^*)y_b)$, $\hat{y}(\sigma^*) = (0, y_b)$. Given the symmetry of the problem, we assume, without loss of generality, that

$$\sigma^* > 1/2.$$

Let

$$\hat{y}_H(\sigma) = (0, \frac{1}{1-\sigma}), \ U_H(\sigma) = (1-\sigma)U(\frac{1}{1-\sigma}),$$

and

$$\bar{\sigma} = \frac{(\alpha-\gamma)(y_b-1)}{y_b(\alpha-\gamma) - -\alpha y_*} \in (0, 1).$$

By $H2$ and $H2'$, $\bar{\sigma} \in (\sigma^*, 1)$. The optimal solution and the value, $W(\sigma)$, of the programming problem, (H^σ), are characterized by the next lemma the proof of which is postponed to the appendix.

Lemma 4. *The optimal solution to (H^σ) is equal to*

$$\hat{y}(\sigma) = \begin{pmatrix} \hat{y}_H(\sigma), \ \textit{for } \sigma \in [1/2, \bar{\sigma}), \\ \hat{y}_H(\bar{\sigma}) \cup (1,1), \ \textit{for } \sigma = \bar{\sigma}, \\ (1,1), \ \textit{for } \sigma \in (\bar{\sigma}, 1]. \end{pmatrix}$$

The value to H^σ, $W(\sigma)$, is equal to $U_H(\sigma)$ and strictly increasing in $[1/2, \sigma^*]$, equal to $U_H(\sigma)$ and strictly decreasing in $[\sigma^*, \bar{\sigma})$, and constant and equal to $U(1,1)$, for $\sigma \in [\bar{\sigma}, 1]$. Furthermore, $W(\sigma^*) > W(\sigma) > U(1,1)$, for $\sigma \in [1/2, \sigma^*)$.

Lemma 4 implicitly characterizes the optimal solution to the programming problem H^J. Given an arbitrary J, let $n(J) \in \arg\max_{\frac{J}{2} \leq n \leq J} W(\frac{n}{J})$. (Remember that, by the symmetry in σ of the programming problem H^σ, $W(\frac{n}{J}) = W(\frac{J-n}{J})$. Hence, the constraint $\frac{J}{2} \leq n \leq J$ is without loss of generality.) By Lemma 4, $\frac{n(J)}{J} \in [\bar{\sigma}, 1]$ only if $\frac{n}{J} \notin [1/2, \bar{\sigma})$, for all $\frac{J}{2} \leq n \leq J$. Evidently, if J is even or sufficiently high, $\frac{n(J)}{J} \in [1/2, \bar{\sigma})$. Let J^* be the smallest odd integer such that $\frac{n}{J} \in (1/2, \bar{\sigma})$ for some $n \leq J$. Then:

Lemma 5. *For $J \geq J^*$, $y(J) = \hat{y}_H(\frac{n(J)}{J})$.*

8.3 Sunspot Equilibria

We construct a family of sunspot equilibrium allocations indexed by J. The construction is basically identical to the one used in the proof of Proposition 9.

For $J > J^*$, partition the interval $[0, 1]$ in J subintervals, $I_j = [j - 1/J, j/J)$, $j = 1, ..., J$. For each j and h, let $h\hat{+}(\frac{j}{J})$ be an addition modulo 1. Then A^J is defined as:

$$A^J(h, j) = y_H(\tfrac{n(J)}{J}) \text{ if } h\hat{+}(\tfrac{j}{J}) > (\tfrac{n(J)}{J}), \text{ while } A^J(h, j) = 0 \text{ if } h\hat{+}(\tfrac{j}{J}) \leq (\tfrac{n(J)}{J}).$$

The allocation map A^J is feasible and $A^J(h) \in y(J)$, for all h. Furthermore, $(\frac{1}{J})\Sigma_j U^J(A(h, j)) = W(\frac{n(J)}{J})$.

Proposition 10. *There exists a countable infinity of Pareto ranked sunspot equilibria.*

Proof. It suffices to show that there exists a sequence of integers J_k such that $W(\frac{n(J_k)}{J_k})$ is monotonically increasing and $W(\frac{n_{J_k}}{J_k}) \rightarrow W(\sigma^*)$. For $J > J^*$, let $n^-(J) \in \arg\max_{\frac{1}{2} \leq \frac{n}{J} \leq \sigma^*}(\frac{n}{J} - \sigma^*)$ and $n^+(J) = \arg\max_{\sigma^* \leq \frac{n}{J} < \bar{\sigma}}(\frac{n}{J} - \sigma^*)$. Evidently, $n^-(J) = n^+(J) - 1$ and $\frac{n(J)}{J} \in \arg\max_{n \in \{n^-(J), n^+(J)\}} W(\frac{n}{J})$.

If σ^* is an irrational number, set $J_k = k$. If σ^* is a rational number, let (n^*, N^*), $n^* \leq N^*$, be the pair of natural numbers satisfying a) $\sigma^* = \frac{n^*}{N^*}$, and b) $N + n > N^* + n^*$, for each pair of natural numbers $(n, N) \neq (n^*, N^*)$ such that $\sigma^* = \frac{n}{N}$. If N^* is even, consider a sequence $J_k = 2k + 1$, and if it is odd $J_k = (2)^k$. In both cases, $\sigma^* - \frac{n(J_k)}{J} \neq 0$, for all k, and $\lim_{k \rightarrow \infty}(\sigma^* - \frac{n(J_k)}{J}) = 0$. By using if necessary subsequences, we get the claim.

Appendix

Proof of Lemma 1 The argument is based on some properties of Young measures [18]. A Young measure on $R_+ \times Y$ is a positive measure τ such that for any Borel set $A \subset R_+$, $\tau(A \times Y) = L(A)$, where L is the Lebesgue measure. The narrow topology on the set of Young measures is defined by the duality of these measures with the Caratheodory's integrands; equivalently a sequence of Young measures converges narrowly if and only if the inner product with any Caratheodory's integrand converges.

Let $\hat{y} : R_+ \to Y$ be a measurable map. The unique Young measure associated to $\hat{y}$, ν, is such that for any function $\hat{U} : R_+ \times Y \to \bar{R}$ measurable and ≥ 0, $\int_{R_+ \times Y} \hat{U} d\nu = \int_{R_+} \hat{U}(t, \hat{y}(t))dt$. Since in our case, Y is compact, $\hat{U} = e^{-rt}U$ and U is continuous, the requirement $\hat{U} \geq 0$ is without loss of generality. A Young measure ν can be represented by its disintegration, which is a family $(\beta_t)_{t \in R_+}$ of probabilities over Y such that for any function $\hat{U} : R_+ \times Y \to \bar{R}$ measurable and non negative, $\int_{R_+ \times Y} \hat{U} d\nu_\omega = \int_{R_+} [\int_Y \hat{U}(t, y)\beta_t(dy)]dt$. If ν is associated to $\hat{y}$, $\beta_t = \delta_{\hat{y}(t)}$, for $\delta_{\hat{y}(t)}$ denoting the Dirac mass at $\hat{y}(t)$. The set of Young measures associated to functions is dense (in the narrow topology) in the set of Young measures. Thus, if (U) has a solution $\hat{y}^* = (\hat{y}^*(\omega, .))_{\omega \in [0,1)}$, the family of Young measures $(\nu_\omega)_{\omega \in [0,1)}$ associated to $\hat{y}$ and, equivalently, its disintegration $\hat{\beta}^*$, $\hat{\beta}^*(\omega, t) = \delta_{\hat{y}^*(\omega,t)}$, is an optimal solution to:

$$(U') \quad \max_{\hat{\beta}(t) \in M_{+,.}([0,1),\hat{B})} \int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{\beta}(\omega, t))d\omega)dt,$$

$$\textit{subject to} \int_{\Re_+} (\int_{[0,1)} \hat{p}(\omega, t)y(\hat{\beta}(\omega, t))d\omega)dt.$$

where $M_{1,+}([0,1), \hat{B}) = \{\hat{\beta}(t) : \Omega \to M_{+,.}(Y) : \beta(t) \text{ is } ([0,1), \hat{B}) - \textit{measurable}\}$.

The argument is concluded by the next claim that creates an obvious parallelism between the programming problems $(U')$and (V).

An optimal solution to (U'), $\hat{\beta}^*$ and to (V), $\hat{y}^*$, are equivalent if

$\int_\Omega V(\hat{y}^*(\omega, t))d\omega = \int_\Omega U(\hat{\beta}^*(\omega, t))d\omega$ and $\hat{y}^*(\omega, t) = y(\hat{\beta}^*(\omega, t))$, for $L \times L - a.e.(\omega, t)$.

Claim. Optimal solution to (U') and to (V) are equivalent.

Proof. This is a trivial consequence of the linearity (in contingent commodities) of $\hat{p}$ and the definition of V.

Proof of Claim 6.2 We break the argument in two steps.

Step 1: The support of the optimal lottery β^* is $\{y_1^*, y_2^*\}$, with $0 < y_1^* < 1 < y_2^* \leq M$.

Proof. By the feasibility condition $y^\theta(\beta^*) = 1$ and by the Inada condition $y_1^* > 0$. Furthermore, as already observed 1 is never in the support of β^*. Thus, the support of β^* contains at least two points. Suppose, by contradiction, that it contains more than two points. Then, there exists at least two points y_1 and y_2 either both strictly less than one or strictly greater than 1. Thus, there exists q^* such that y_1 and $y_2 \in \arg\max_y y^\alpha - q^* y^\theta$ or y_1 and $y_2 \in \arg\max_y y - q^* y^\theta$. However, both functions $y^\alpha - q^* y^\theta$ and $y - q^* y^\theta$ are strictly concave and therefore they have a unique maximizer.

Step 2: The intertemporal economy $(U, y^\theta, 1)$ has a unique efficient allocation $(\hat{x}, \hat{a}, \hat{\beta})$, $\hat{x}(t) = 1$, $\hat{a}(t) = 0$ and $\hat{\beta}(t) = \beta^*$, for $L - a.e.t.$

Proof. We just prove that there exists a unique stationary allocation. A minor modification of the argument implies the claim. By the form of the technology, the optimal stock of capital is $x = 1$. By contradiction, suppose that there exists two distinct optimal lotteries β and γ with supports $\{y_{1,\sigma}, y_{2,\sigma}\}$, $y_{1,\sigma} < 1 < y_{2,\sigma} \leq M$. $\sigma = \beta, \gamma$. Denote with σ, in addition to the optimal lottery, the probability of $y_{1,\sigma}$, $\sigma = \alpha, \beta$. Since, $y^\theta(\sigma) = 1$, $\sigma = \beta, \gamma$, and $U(\beta) = U(\gamma)$, $\{y_{1,\beta}, y_{2,\beta}\} \neq \{y_{1,\gamma}, y_{2,\gamma}\}$, otherwise, the two lotteries are identical. Define the lottery η which assigns probability $\eta \equiv \frac{\beta+\gamma}{2}$ to the allocation $y_1(\eta) \equiv \frac{\gamma y_{1,\gamma} + \beta y_{1,\beta}}{\beta+\gamma}$ and probability $1 - \eta = \frac{2-(\beta+\gamma)}{2}$ to the allocation $y_2(\eta) \equiv \frac{(1-\gamma)y_{2,\gamma} + (1-\beta)y_{1,\beta}}{2-(\gamma+\beta)}$. The lottery η is feasible since by convexity:

$1 = \frac{1}{2}\{y^\theta(\beta) + y^\theta(\gamma)\} = \frac{\gamma+\beta}{2}((\frac{\beta}{\beta+\gamma}(y_{1,\beta})^\theta + \frac{\gamma}{\beta+\gamma}(y_{1,\gamma})^\theta) + \frac{2-(\gamma+\beta)}{2}(\frac{1-\beta}{2-(\gamma+\beta)}(y_{2,\beta})^\theta + \frac{1-\gamma}{2-(\gamma+\beta)}(y_{2,\gamma})^\theta)] > \eta(y_{1,\eta})^\theta + (1-\eta)(y_{2,\eta})^\theta = y^\theta(\eta)$.

Furthermore, by concavity:

$U(\eta) = \frac{\gamma+\beta}{2}(\frac{\beta}{\beta+\gamma}y_{1,\beta} + \frac{\gamma}{\beta+\gamma}y_{1,\gamma})^\alpha + \frac{2-(\gamma+\beta)}{2}(\frac{1-\beta}{2-(\gamma+\beta)}y_{2,\beta} + \frac{1-\gamma}{2-(\gamma+\beta)}y_{2,\gamma})^\alpha] \geq \frac{1}{2}[y^\alpha(\beta) + y^\alpha(\gamma)] = U(\gamma) = U(\beta)$.

Thus, the lottery η is weakly preferred to the optimal lotteries β and γ and frees up resources. Thus, $y_1(\eta)$ can be increased (by an arbitrarily small amount) delivering a lottery that Pareto dominates the optimal ones. A contradiction.

The Efficient Stationary Allocation of Example 5

Bear in mind that $f(x_i^*) = x_i^* = 1$, for $i = 1, 2$. Since U and F are separable, $y_2^* = f(x_1^*) = 1 \in \arg\max_{y_2} y_2 - q_2^* y_2$, for $q_2^* = 1$.

For given $q_1 > 0$, let $Y_1(q_1)$ be the set of optimal solution to:

$$\max_{y_1} u(y_1) - (q_1/2)(y_1)^2, \tag{30}$$

To pin down the efficient allocation (for the first good) , we have to find a price $q_1 > 0$ and a feasible lottery β over the $Y_1(q_1)$, i.e., a lottery that satisfies $\int_{Y_1(q_1)} y_1^2 \beta(dy_1) = 1$.

Lemma 6. *The unique feasible and optimal solution to 30 is, for $q^* = 1$, $y_1^* = 1$.*

At this point, for sake of clarity, we break the argument in two steps. First we show that, by the feasibility conditions, $q^* = 1$. Then we show that for $q_1^* = 1$, the optimal solution is $y_1^* = 1$.

Step 1: $q_1^* = 1$.

Proof. Suppose by contradiction that $q_1^* \neq 1$. If an optimal solution $y_1 \neq 1 \pm \varepsilon$ exists, it must satisfy the following first order conditions:

$$(11/1-\varepsilon) - q_1^* y_1 = 0, \qquad y_1 \in [0, 1-\varepsilon)$$
$$1 - q_1^* y_1 = 0, \qquad y_1 > 1+\varepsilon$$
$$1 + \alpha(y_1 - 1) - q_1^* y_1 = 0, \qquad y_1 \in (1-\varepsilon, 1+\varepsilon).$$

The first order conditions are satisfied by $y_1 = (1/q_1^*) > 1 + \varepsilon$, if $q_1^* < \alpha$, and by some $y_1 > 1$, for $q \in [\alpha, 1)$. Furthermore, for $q < 1$, $y_1 = 1 - \varepsilon$ can not be an optimal solution, since $\lim_{y \to (1-\varepsilon)^-} [u'(y_1) - q(y_1)\} > 0$ and $\lim_{y \to (1-\varepsilon)^+} [u'(y_1) - q(y_1)] = 1 - q_1^* + \varepsilon(q_1^* - \alpha) > 0$. Hence, if $q < 1$, the consumption bundles that maximize the problem 30 are all greater than 1, thereby violating the feasibility conditions. Hence, it must be $1 \leq q_1^*$. However, if $q_1^* > 1$, the first order conditions imply that $y_1^* < 1$. Furthermore, for $q > 1$, $y_1 = 1 + \varepsilon$ can not be an optimal solution, since $\lim_{c \to (1+\varepsilon)^-} [u'(c_1) - q(c_1)] = 1 - q_1^* - \varepsilon(q_1^* - \alpha) < 0$ and $\lim_{c \to (1+\varepsilon)^+} [u'(c_1) - q(c_1)\} < 0$. Once again, feasibility conditions are violated. Hence, $q = 1$.

Step 2: $y_1^* = 1$

Proof. Observe that, for $q_1^* = 1$,

a) $y_1^* = 1$ satisfies the first and second order conditions

b) it does not exists y_1, $y_1 \neq 1 - \varepsilon$ and $y_1 \neq 1 + \varepsilon$, that satisfies the first order conditions.

However, at $y_1 = 1 - \varepsilon$ and at $y_1 = 1 + \varepsilon$, the objective function $u(y_1) - (y_1)^2/2$ is not differentiable, but it is locally concave. Hence, in order to rule out $1 - \varepsilon$ and $1 + \varepsilon$ as possible optimal solutions we compare the values of the objective function at 1, $1 + \varepsilon$ and $1 - \varepsilon$. By direct computation:

$[u(1-\varepsilon) - (1-\varepsilon)^2/2] - [u(1) - (1)^2/2] = -(1/2)(1-\alpha)\varepsilon^2 < 0,$

and

$[u(1+\varepsilon) - (1+\varepsilon)^2/2] - [u(1) - (1)^2/2] = -(1/2)(1-\alpha)\varepsilon^2 < 0.$

Hence, the claim.

Proof of Lemma 2. By contradiction, suppose that there exists $(q, \hat{p})$, such that $\hat{y}^*$ is (part of) a static sunspot equilibrium. Bear in mind that $\hat{y}_1^*(\omega) = 1$, $\hat{p}_2(\omega)\hat{y}_2^*(\omega) = q_2 \hat{y}_2^*(\omega)$, $L - a.e.\omega$, and $u'(1) = 1$. Then, by the first order conditions of the individual problem (H):

$u'(\hat{y}_1^*(\omega)) = u'(1) = 1 = \eta \hat{p}_1(\omega)$, $1 \leq \eta \hat{p}_2(\omega)$ and $\hat{y}_2^*(\omega) = \eta \hat{p}_2(\omega) \hat{y}_2^*(\omega)$, for some $\eta > 0$, $L - a.e.\omega$.

Then, last conditions and equations 22 immediately imply that:

i) $\hat{p}_1(\omega) = 1/\eta$, $L - a.e.\omega$, and $q_2 = 1/\eta$;

ii) $\int_{[0,1)} \hat{p}_2(\omega)\hat{y}_2^*(\omega) d\omega = q_2 \int_{[0,1)} \hat{y}_2^*(\omega) d\omega = \int_{[0,1)} (\hat{y}_2^*(\omega)/\eta) d\omega$, and

Then

$$\int_{[0,1)} \hat{p}(\omega)\hat{y}^*(\omega)d\omega = (q_2, q_2)\int_{[0,1)} \hat{y}^*(\omega)d\omega = 2q_2 = 1 + q_2, \text{ i.e., } q_2 = 1$$

Consider the constant allocation $\hat{y}'$ defined by $\hat{y}'(\omega) = (2,0)$ for all ω. Evidently, $\hat{y}'$ is budget feasible at (q,p) and $\int_{[0,1)} U(\hat{y}'(\omega))L(d\omega) = 2$. However: $2 > \int_{[0,1)} U(\hat{y}^*(\omega))d\omega = u(1) + \int_{[0,1)} \hat{y}_2(\omega)d\omega = 1 + u(1) = 2 - (1/2)\alpha\varepsilon^2$ ∎

Proof of Lemma 3 This is consequence of Jensen's inequality. More precisely, the utility function U is concave in $[0, y_*]$ and in $[y_*, \infty)$. For each $y^J \in y^J(J)$, let $y^{J_1} = (y_1, ..., y_{J_1})$ be the vector of components of y^J contained in the interval $[0, y_*]$, i.e., $y^{J_1} \in [0, y_*]^{J_1}$. There is no loss of generality in assuming that the entries of y^{J_1} are the first J_1 entries of y^J. Let $J_2 = J - J_1$. Evidently, by construction, $y^{J_2} = (y_{J_1+1}, ..., y_J) \in (y_*, \infty)^{J_2}$. Then:

$$\Sigma_j U(y_j) = J_1 \Sigma_{j=1}^{J_1} \frac{U(y_j)}{J_1} + J_2 \Sigma_{j=J_1+1}^{J} \frac{U(y_j)}{J_2} \leq$$

$$[(J_1 U(\Sigma_{j=1}^{J_1}(y_j/J_1)) + J_2 U(\Sigma_{j=J_1+1}^{J}(y_j/J_2)].$$

∎

Proof of Lemma 4 Given the shape of U, for $\sigma \in [1/2, 1)$, the optimal solutions to (U^σ) is either i) the constant bundle $\bar{1} = (1,1)$ and/or ii) $\hat{y} \neq \bar{1}$, with $\min\{y_1, y_2\} = 0$ and $\sigma\hat{y}_1 + (1-\sigma)\hat{y}_2 = 1$.

There are just two candidates that satisfy *ii*). They are obtained by setting equal to 0 either the first or the second entry of the consumption bundle. The first candidate $\hat{y}_H(\sigma)$ has already been defined, the second, $\hat{y}_L(\sigma)$, is defined as follows:

$$\hat{y}_L(\sigma) = (\frac{1}{\sigma}, 0) \text{ and } U_L(\sigma) = \sigma U(\frac{1}{\sigma})$$

Let $\Phi(\sigma) = U_H(\sigma) - U_L(\sigma)$. For $\sigma \in [\sigma^*, 1)$, $\frac{1}{1-\sigma} \geq y_b > \frac{1}{\sigma}$, and hence:

$$\Phi(\sigma) = (1-\sigma)[(\alpha - \gamma)y_b - \gamma y_*] + \gamma - \alpha(1 - \sigma y_*).$$

Then:

$$\Phi(\sigma^*) = (\alpha\sigma^* - (1-\sigma^*)\gamma)y_*$$

$$\Phi(1) = \gamma - \alpha(1 - y_*).$$

Since $\sigma^* > 1/2$ and $\alpha > \gamma$, $\Phi(\sigma^*) > 0$. Furthermore, by $U3$, $\Phi(1) > 0$, Hence:

$$U_H(\sigma) > U_L(\sigma), \ for \ \sigma \in [\sigma^*, 1),$$

For $\sigma \in (1/2, \sigma^*]$, $\frac{1}{1-\sigma} > \frac{1}{\sigma} > y_*$ and, by direct computation:

$$\Phi(\sigma) = \alpha(1 + (2\sigma - 1)y_*) > 0$$

Hence, $U_H(\sigma) > U_L(\sigma)$, for all $\sigma \in [1/2, 1)$. Furthermore:

$$U_H(\sigma) - U(\bar{1}) = \left\{ \begin{array}{c} \alpha\sigma y_*, \text{ for } \sigma \in [1/2, \sigma^*) \\ (\alpha-\gamma)((1-\sigma)y_b - 1) + \alpha\sigma y_*, \text{ for } \sigma \in [\sigma^*, 1) \end{array} \right\}.$$

Therefore, by trivial computations, $U_H(\bar{\sigma}) = U(\bar{1})$, $W(\sigma) = U_H(\sigma)$, for $\sigma \in [1/2, \bar{\sigma})$, while $W(\sigma) = U(\bar{1})$, $\sigma \in [\bar{\sigma}, 1)$. Finally, by taking into account assumption $H2$),

$$dW(\sigma)/d\sigma = \left\{ \begin{array}{c} \alpha y_* > 0, \text{ for } \sigma \in [1/2, \sigma^*) \\ -(\alpha-\gamma)y_b + \alpha y_* < 0, \text{ for } \sigma \in (\sigma^*, \bar{\sigma}) \\ 0, \text{ for } \sigma \in [\bar{\sigma}, 1) \end{array} \right\}.$$

■

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Shaking the Tree: An Agency-Theoretic Model of Asset Pricing*

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1 Introduction

The conditional heteroskedasticity and long-memory persistence apparent in financial times series is one of the most thoroughly studied but least understood of empirical regularities in financial economics. These regularities were first noted formally in the mid-1960s by researchers examining the long-run statistical behavior of stock prices, interest rates, and foreign exchange rates (see e.g. [8], [12], or [21]). In the early 1980's, the availability of high-speed computers made it possible for researchers to begin modeling the time-varying behavior of these time series explicitly. Early work on these topics includes the original Autoregressive Conditional Heteroskedasticity (ARCH) model of [7], and work by [17] and by [10] on fractional differencing (which built on the seminal analysis of fractional Brownian motions by [22]). Since that time, a number of extensions and elaborations of these models have appeared (see [2] for a detailed review of this literature). Application of these models to the stock price, interest rate, and foreign exchange data generally yields results which are highly statistically significant. It is no surprise, then, that ARCH and fractionally integrated time-series models have become increasingly more popular as tools for analyzing financial data.

What is surprising, however, is the relative dearth of theoretical results which might explain the observed time-varying volatility and/or persistence in the data. Standard asset pricing models of the type originally formulated by [20] or [3] are simply too stationary to deliver the desired effects. In the [20] framework, for example, the price of asset i is given by

$$p_i(y_t) = \beta E_t \left\{ M_i(y_t, y_{t+1}) \left(y_{t+1} + p_i(y_{t+1}) \right) \mid y_t \right\}$$

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where β is the discount rate, M_i is the pricing kernel, and y_t is the vector of dividends paid at time t. With a stationary pricing kernel, only the assumption of conditional heteroskedasticity in the dividend process will deliver ARCH effects in this framework. In models with production, convergence to steady-state equilibrium has the effect of eliminating non-stationarities in the model which might generate time-varying conditional volatilities, unless, of course, one assumes that the exogenously given shock process is itself ARCH. This approach has been examined explicitly in work by [1] and [11]. Each of these papers allows for underlying dividend processes with time-varying conditional variances and examines the implications of changing risk on the general equilibrium behavior of the model. While such studies can yield interesting insights, they hardly constitute a theoretical explanation of ARCH.

There have been several hypotheses put forth to explain ARCH effects. One such hypothesis holds that serially correlated news arrival drives increases in variance (see, for example, [5] or [9] for details). While this is plausible on its face, is poses the obvious question as to why the information arrival process should be serially correlated. Other researchers ([26], [27]) have examined time deformations that can occur when calendar time and market time proceed at different rates. In these models, trade can occur more or less frequently in the same calendar period. This can lead to observed volatilities which vary with calendar time even though the asset is covariance stationary when measured in "operational time". While these models do exhibit the desired conditional heteroskedasticity, they do not explain what might drive the fluctuations in trading rates that give rise to the time deformation to begin with.

A third hypothesis has been put forward by [14] to explain the observed ARCH effects in interest rate time series. In their model, market incompleteness together with constraints on borrowing leads to differences in savings behavior between rich and poor agents. This implies that the distribution of wealth in the economy influences the degree to which otherwise nicely behaved stochastic endowment shocks affect the interest rate in the model. Time variation in the distribution of income translates into time variation in the volatility of interest rates. While this mechanism may be at work in generating ARCH effects in interest rates, it seems implausible that variations in the wealth distribution could occur with sufficient frequency to drive the ARCH effects observed in stock prices.

The literature on theoretical explanations of the observed persistence in economics time-series is even sparser. The earliest result is work by [13] showing how aggregation of independent AR(1) processes can generate a process which is fractionally integrated. Since many financial and economic decision problems in stochastic environments lead to decision rules and pricing relations which are autoregressive, this would seem a plausible explanation for why economic aggregates exhibit persistence. One problem with this result, however, is that it is driven by the requirement that there exist AR(1) processes which have autoregressive parameters very close to 1. This requirement is generally not consistent with observed microeconomic data. A second problem is that the result isn't applicable to much of the financial data which is available in disaggregate form. More recent work on this issue has been done by [6] who examines a model based on the Ising models of physics. In Durlauf's model, firms are located on a lattice, and the activities of neighboring firms "spill over" and influence the original firm's production activities.

In this paper, we examine a very different mechanism based on an agency theoretic model of the internal dynamics of the firm. One of the weaknesses of exchange models of asset pricing is the need to exogenously specify the dividend process. This weakness is remedied somewhat in models of capital accumulation, since production is modeled explicitly. But the neoclassical view of the firm as a shell housing the basic technical processes that transform inputs into outputs fails to capture several essential aspects of real firms that may be relevant in determining the value of the firm. A key feature, which we focus on here, is the ability of a firm's managers to respond to favorable opportunities by expanding output (or sales from inventory) and to reduce output (or restock inventory) in response to unfavorable shocks.

We examine this feature using a standard principal-agent model together with a particular reduced form for production which allows the agent to control, at some cost, the rate of growth of the firm's output. To keep the analysis relatively simple, we embed this part of the model in a simple variant of the [20] asset-pricing model. As in Lucas, we have a single representative agent who owns an orchard full of trees (the assets) which bear stochastic amounts of completely perishable fruit (dividends) in every period. Unlike Lucas, however, we do not assume that the process of fruit production is completely exogenous. Instead, we include an agent, who one may think of as the gardener, who can influence the production of fruit by exerting effort in the orchard. Specifically, we assume that the agent can cause the amount of fruit produced to grow by exerting effort. In the context of the tree model, we can think of our gardener's fixed amount of time being allocated between routine crop tending and innovations in fertilizers, root stocks, and hybridization. The latter facilitate growth, which in turn demands more of the former activities, making it more difficult to engage in activities that promote innovation.

In order to keep the analysis relatively simple, we will focus on the case of a single asset or firm. The owner of the firm contracts with a manager to operate the firm and pays him a portion of the output as compensation. Again, for simplicity, we assume that the principal's interest is in maximizing the value of the firm, while the agent is risk averse with respect to his income. We consider a simple repeated relationship between the principal and agent without commitment. Under these assumptions, we show that the optimal contract generates a time-series for the firm's price which exhibits significant conditional heteroskedasticity and long-memory persistence, even when the underlying innovations are i.i.d.

In Section 2 we lay out the formal model. In Section 3 we look at a specific parametrization of the model and indicate how to solve this model numerically. Section 4 compares the first-best and second-best contract equilibria for the model, using both analytical conclusions and the numerical solution. Section 5 examines simulated time-series data generated by the model, and tests the data for evidence of long-memory persistence and of ARCH effects. We find that for certain specifications of the manager's preferences, both of these phenomena occur. Section 6 presents conclusions, while an Appendix contains the solution of the first-best contract, as well as the existence proof for the second-best contract equilibrium and details of the numerical simulation.

2 The Model

The model is based on the stochastic asset pricing model of [20], in which the firm is comprised of two agents, an owner and a manager. The manager's action is defined as an effort level which contributes to the production of the firm. Following Lucas, we associate the firm's production with a simple specification of a dividend process:

$$x_t \sim G(x_t \,|\, x_{t-1}, a_t) \tag{1}$$

where x_t and x_{t-1} are the current and previous dividends, respectively, a_t is the manager's action, and G is a continuous conditional distribution function over a compact support X—for simplicity we specify a first order dividend process.

Each period the manager chooses an action which maximizes the expected utility for the upcoming period. We assume that the manager is accepting a series of one-period contracts, and cannot commit to a longer contract. The manager's preferences are separable and given by

$$U(w_t, a_t) \equiv u(w_t) - h(a_t) \tag{2}$$

where w_t is the manager's wealth, defined as the wage paid by the owner, u is an increasing, continuous concave function, and h is a strictly increasing, continuous convex function. The h function measures the disutility of working to the manager. Finally, we assume that the agent possesses outside employment opportunities, which sets a lower bound $\bar{u}$ for the utility the agent must receive from employment.

The owner of the firm wishes to maximize the firm's value, i.e. the asset price. The owner must pay the manager a wage w_t in return for the manager's labor a_t in production (which is summarized by the dividend process). We suppose that the manager's labor contribution is unobserved when the wage is specified–thus the owner can only condition the wage upon the current dividend x_t. The owner seeks to

$$\max_{a_t, w_t} E_{t-1}(p_t), \tag{3}$$

where E_j denotes the conditional expectation operator with respect to information known at time j.

As in [20] the price of the firm is given by the discounted expected future dividends, net of the labor wage. This can be succinctly written as

$$p_t = \beta E_t p_{t+1} + x_t - w_t(x_t), \tag{4}$$

where $\beta \in (0,1)$ is the owner's discount factor.

To complete the model, some mechanism for the owner's expectation of the future price must be defined. Suppose that the owner uses prior values of the state variables, in this case the dividend, in order to form her expectations. We assume that only the expected current dividend value is used in the forecast, and that the forecast function $v : X \to \mathbb{R}_+$ is continuous and time invariant:

$$E_t(p_{t+1}) = v(x_t). \tag{5}$$

The owner's problem is to select an action a_t and a wage w_t to maximize the value of the firm, given that the action of the agent will be unobservable. Using equations (1) to (5) we can restate the owner's problem as

$$\max_{a_t, w_t} \int_X (x - w_t(x))\, dG(x\,|x_{t-1},\, a_t) + \beta \int_X v(x)\, dG(x\,|x_{t-1},\, a_t) \tag{6}$$

such that

$$a_t \in \arg\max_a \int_X u(w_t) dG(x \mid x_{t-1}, a) - h(a), \tag{7}$$

$$\int_X u(w_t)\, dG(x\,|x_{t-1},\, a_t)\; -\; h(a_t) \geq \bar{u}.$$

Note that in this form the owner faces a typical second-best repeated static moral hazard problem–the dividend process is conditioned on the manager's choice of labor, but this is not incorporated into the contracting institution by the agents. While the single-period contract specification is used for reasons of tractability, it may be helpful to think of this as describing a firm which hires and fires labor of the same type every period. A newly-hired manager may observe that previous labor has affected the dividend process (as the distribution function for dividends is common knowledge) but has no control over the previous manager's labor choice.

We would like to examine the solution to the owner's problem without having to worry about the learning dynamics associated with the forecast function v. That is, we will assume that the owner has already learned the rational expectations equilibrium (REE) price function, and uses this function to forecast future prices. In other words, the owner knows the actual function v^* which takes the observed dividend and returns the expected value of the firm, i.e. $E_{t-1} p_t \equiv v^*(x_{t-1})$. This means that the REE price function must satisfy

$$v^*(x_{t-1}) = \max_{a_t, w_t} \left\{ E_{t-1}\left[x_t\,|x_{t-1}, a_t\right] - \int_X w_t(x)\, dG(x\,|x_{t-1},\, a_t) \right. \tag{8}$$
$$\left. + \beta \int_X v^*(x)\, dG(x\,|x_{t-1},\, a_t) \right\}.$$

such that

$$a_t \in \arg\max_a \int_X u(w_t)\, dG(x\,|x_{t-1},\, a) \;-\; h(a), \tag{9}$$

$$\int_X u(w_t)\, dG(x\,|x_{t-1},\, a_t) \;-\; h(a_t) \geq \bar{u}. \tag{10}$$

In the Appendix it is shown that under the current assumptions on the dividend process and some additional restrictions on preferences, there exists a function v^* which defines at each point in time the value of the firm. However, in order to more fully characterize the qualities of the value function (and the associated optimal labor and wage functions, respectively) it is necessary to introduce functional form assumptions on the dividend process and on preferences. These assumptions also allow for numerical simulations to take place. In selecting these functional forms we have attempted to maintain a balance between generality and tractability–even with the specifications given below, the rational expectations price function is still general enough that an analytical solution cannot be found, and numerical analysis must be performed.

Definition 1. *The dividend process is an AR(1) process with innovation in the mean, i.e.*

$$G(x_t\,|\,a_t, x_{t-1}) \Leftrightarrow x_t = a_t + \rho x_{t-1} + \varepsilon_t,\; |\rho| < 1,\; \varepsilon_t \sim N(0,1)\,\forall t.$$

Definition 2. $U(w_t, a_t) \equiv -\exp(-w_t) - \exp(a_t)$.

With Definition 2 we may adopt the first-order-approach (see e.g. [25], [18]) to replace the argmax operator in equation (7) with the associated first-order condition:

$$\frac{\partial}{\partial a} \int_X u(w_t(x))\, dG(x\,|\,a, x_{t-1}) - h(a)\bigg|_{a=a_t} = 0. \tag{11}$$

3 Numerical Approximation

The specifications for the dividend process and preferences are not enough to generate an analytical solution for the value, wage or labor functions. The approach taken in this paper is to leave the remaining functional forms as general as possible, and to instead focus on numerical solutions to the REE condition. The aim here is to identify and analyze the dynamics of the price and dividend processes given the numerical solution, instead of concentrating solely upon the analytical results of e.g. a local quadratic approximation of the value function v^*.

However, we would like to be able to compare the resulting owner-manager contract and, ultimately, the dynamics of the price process with a benchmark case. We can then see what this type of environment is 'bringing to the table' when compared with other contracting forms. In particular, it is interesting to compare the 'second-best' model outlined above with the 'first-best' problem, in which the manager's action is set by the owner without considering the manager's optimal choice. In this case the incentive compatibility condition is absent from the owner's optimization, and the only thing the owner need worry about is giving the manager enough utility (in this case $\bar{u}$) to choose employment.

In the first-best case, the problem facing the owner is

$$\max_{a_t, w_t} \left\{ E_{t-1}\left[x_t \,|x_{t-1}, a_t\right] - \int_X w_t(x)\, dG(x\,|x_{t-1},\, a_t) + \beta \int_X v(x)\, dG(x\,|x_{t-1},\, a_t)) \right\}$$

such that

$$\int_X u(w_t)\, dG(x\,|x_{t-1},\, a_t) \;-\; h(a_t) \geq \bar{u}.$$

In the Appendix it is shown that when Definitions 1 and 2 hold the wage function has the usual property that the manager's risk is entirely smoothed away–regardless of the observed dividend, he always receives a utility of $\bar{u}$. Both the wage function and the effort function are constant, and the expected price function v^* is linear and increasing. Further details on the first-best contract will be presented when compared with the second-best contract below.

Unfortunately, in the second-best case it is not possible to find a closed-form solution. So there remains the problem of finding the expected price function (or 'value function') v^* of the firm. We adopt here a numerical approximation technique to identify the value and policy functions. The method of obtaining the value function used here is by iterating on a functional operator T, defined by

$$T\left(v^n\right)(x_{t-1}) = E_{t-1}\left[x_t \,|x_{t-1}, a_t^n\right] - \int_X w_t^n(x)\, dG(x\,|x_{t-1},\, a_t^n) + \beta \int_X v^n(x)\, dG(x\,|x_{t-1},\, a_t^n). \tag{12}$$

where w_t^n and a_t^n are the optimal choices given the candidate value function v^n. Since T is a contraction (see Appendix) it follows that

$$\lim_{n\to\infty} T^n(v_0) = v^* \,\forall v_0 \in C_X^1. \tag{13}$$

This condition simply states that for any initial continuous (bounded) function taking dividends into prices, iterations of the operator T on the initial function will converge to the rational expectations price function.

In the approximations these iterations are derived from a class of functions known as universal approximators. A universal approximator has the property that given a finite collection of points from the domain and range of an unknown function, the approximator can update its parameters such that it converges almost everywhere to the unknown function. Members of the class of universal approximators include neural networks, which specify a 'general' functional form up to a finite set of parameters. These parameters are then updated by iteration using the collection of points from the unknown function until they approach a set of 'true' parameters which in principle allow the neural network to arbitrarily approximate the unknown function. (See [16] for a discussion of universal approximation and neural networks, and [29], [19] for a general discussion of the applicability of neural networks to both functional approximation and regression analysis.) The properties of neural networks are by now well established–for the purpose of this paper, they are essentially a convenient method of implementing nonlinear least squares regression in a deterministic setting.

We are now in a position to outline the algorithm for numerically computing the rational expectations value function v^*:

1. Select an initial value function v^0. Using neural networks, this amounts to defining a network whose parameters are randomized.
2. Specify a finite grid over the dividend space X. Given v^0 and a distribution for the error process ε_t, numerically compute for each point in the grid the optimal values for w_t^0 and a_t^0 (i.e. carry out the optimization on the right-hand side of equation 12, where $n = 0$). For each grid point, the value for $T \circ v^0$ is computed.
3. Iterate the value function to $v^1 = T \circ v^0$, for which there is a finite collection of values given by step 2. These values define a neural network f^1 which approximates the function v^1.
4. Use the neural network approximation f^1 in place of v^0 in step 2. Repeat steps 2-3 until $\left\| f^n - f^{n-1} \right\| < \eta$, where η is some predefined error tolerance level. Call f^n the rational expectations value function, or v^*.
5. Given each point in the dividend grid, perform the optimization on the RHS of equation (12) using v^*. This gives in the optimal values for the wage and effort level for each grid point. These values serve to define a neural network $[w_t^*, a_t^*] = [w^*(x_t, x_{t-1}), a^*(x_{t-1})]$, which is the 'policy function' for the economy. Given the state of the economy (i.e., a realization of the current dividend, plus the previous dividend) the policy function tells the principal what wage should be paid, and how hard the manager should work in the current period.

The rational expectations value function is the law of motion for the economy. Once found, v^* defines the optimal a_t^*–this, combined with the previous level of the dividend x_{t-1} and a new realization of the error process ε_t, generates the current dividend x_t. Having observed x_t, v^* also defines w_t^*–this is the wage paid to the agent by the owner and depends only upon $[x_t, x_{t-1}]$ since the owner cannot observe the manager's effort. The actual price of the asset p_t is then

$$p_t = \beta v^*(x_t) + x_t - w^*(x_t, x_{t-1}). \tag{14}$$

With the current dividend x_t the next effort level a^*_{t+1} can be defined, and the process continues. Thus, the economy can be simulated and sequences of dividends and prices can be generated and analyzed.

Particulars on the specification of the networks $v^*(x_{t-1})$ and $[w^*(x_t, x_{t-1}), a^*(x_{t-1})]$, including the number of hidden units, number of iterations, convergence criteria, and other parameter values may be found in the Appendix.

4 First-Best and Second-Best Comparison

We start with the analytical 'benchmark' results for the first-best case. In the Appendix it is shown that the optimal value and policy functions are:

$$E_{t-1}p_t = v^*(x_{t-1}) = \frac{1}{1-\beta}\left(\frac{a^*}{1-\beta\rho} + \ln(|\overline{u}| - e^{a^*})\right) + \frac{\rho}{1-\beta\rho}x_{t-1}, \quad (15)$$

$$a^* = \ln\left(\frac{|\overline{u}|}{2-\beta\rho}\right),$$

$$w^* = -\ln\left(|\overline{u}|\left(\frac{1-\beta\rho}{2-\beta\rho}\right)\right).$$

Figure 1 displays the optimal value function for the case where $\bar{u} = -2$, $\beta = 0.9$, and $\rho = 0.9$. These are the identical parameter values used for the second-best approximation.

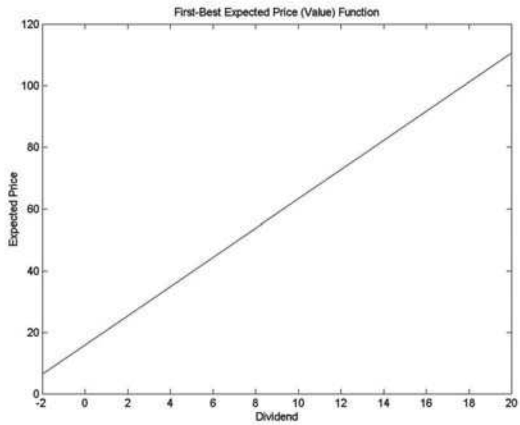

Fig. 1. First-Best Contract Expected Price

Clearly, in the first best case we will have no conditional heteroskedasticity or long memory–the actual price follows the process

$$p_t = x_t + \ln\left(|\overline{u}|\left(\frac{1-\beta\rho}{2-\beta\rho}\right)\right) + \beta\frac{1}{1-\beta}\left(\frac{a^*}{1-\beta\rho} + \ln(|\overline{u}| - e^{a^*})\right) + \frac{\rho}{1-\beta\rho}x_t \Leftrightarrow$$

$$p_t = c + \frac{1-\beta\rho+\rho}{1-\beta\rho}x_t, \tag{16}$$

where we have grouped constant terms into c for convenience. Since

$$x_t = a^* + \rho x_{t-1} + \varepsilon_t$$

we can lag and substitute (16) into the dividend relation to yield

$$p_t = \rho p_{t-1} + (1-\rho)c + \left(\frac{1-\beta\rho+\rho}{1-\beta\rho}\right)(a^* + \varepsilon_t).$$

The price thus follows an AR(1) process with the same autoregressive parameter as the dividend process, but with a different mean and variance. This result supports the intuition that because in the first-best case there is no response by the agent to changes in production, there should be no resultant correlation between production (or price) volatility from period to period. In the first-best case, the sole connection between the present and the past is the autoregressive dividend process.

In the second-best case, however, things are markedly different. Figures 2-4 present the numerical results for the second-best value function, wage function and effort function respectively. Figure 2 shows the numerical approximation of the value function given by the program (8)-(10), using Definitions 1 and 2 and the parameter values $\bar{u} = -2$, $\beta = 0.9$, and $\rho = 0.9$. The value function is strictly concave, and for high levels of the dividend the expected future price is nearly constant. This reflects the fact that for very high levels of the dividend the manager shirks a great deal, absorbing any expected future gains from the dividend. Figure 3 presents the manager's effort function–note that for low or negative dividend levels the manager wishes to work a (small) positive amount, but that as dividends rise the manager rapidly works less and less. Figure 4 presents the optimal wage as a function of the current and past dividends. From this we see that the manager receives positive compensation when there is a large positive gain in the dividend process (indicating hard work by the agent). Compensation then falls as the difference between the two dividends falls, and becomes sharply negative when the current dividend is far below the previous one.

5 Simulation of Time Series

Once the value function and optimal wage and effort functions have been approximated it is possible to simulate the economy and generate synthetic time series for analysis. The simulation of time series data is important because the second-best model has no analytical solution. Thus, analytical tools such

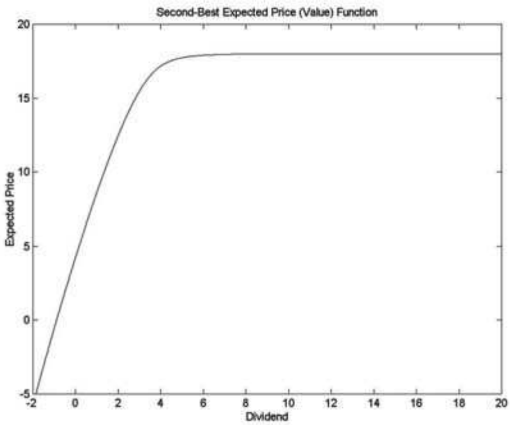

Fig. 2. Second-Best Contract Expected Price

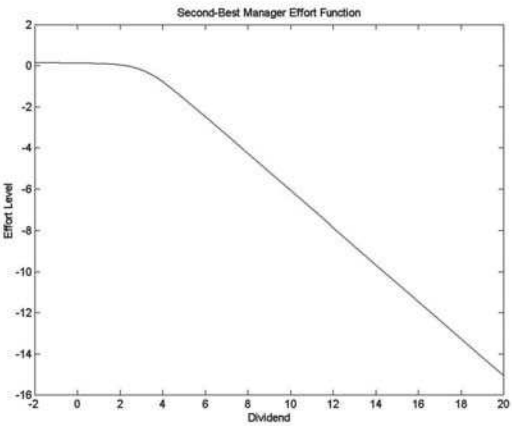

Fig. 3. Second-Best Contract Manager Effort

as comparative statics must be traded for tools which take advantage of the large number of simulated data points that can be generated from the model. Tools such as regression analysis do, of course, make the tacit assumption that the model is an accurate representation of the salient features of the owner-manager relationship in a real economy. Nonetheless, the complexity of the model and the relative paucity of relevant data in the real world are both strong incentives to use simulated time series data as a proxy for economic data generated from actual owner-manager behavior.

The owner-manager model is by its nature a nonlinear model of pricing. Thus, one might expect the dynamics of observed price and dividend sequences to reflect this nonlinearity. Of particular interest is the extent to

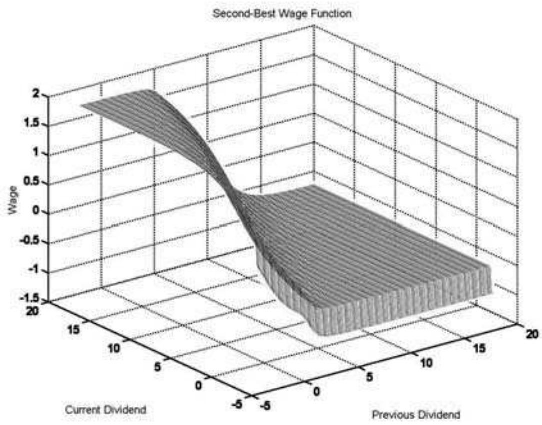

Fig. 4. Second-Best Contract Wage

which the nonlinearity in the model is capable of generating time-series data with properties observed in real stock price data. Thus, in analyzing the simulated time-series, we look for evidence of conditional heteroskedasticity and long-memory persistence.

The tools used for the analysis of the simulated time series are the generalized ARCH (or GARCH) model of conditional heteroskedasticity (see [7] and [2]), the estimated power spectrum of the simulated time-series, and a test for long-memory persistence. The results of this analysis indicate that 1) the price series exhibits strong ARCH-like behavior, with the GARCH (1,1) model demonstrating significant correlated volatility, and 2) the power spectrum and the long-memory test indicate some long-memory persistence.

The second-best value function and the optimal wage and action functions were used to generate time series of both prices and dividends for one hundred thousand periods. A typical example of the dividend time series, along with the associated wage compensation and effort levels for 200 periods are given in Figures 5 and 6. A sample path of the second-best price for the same 200 periods is presented in Figure 7. Since it is the value of the firm which empirically demonstrates ARCH-like behavior and long-memory persistence, only the time series for the price of the firm was tested in what follows.

5.1 Spectral Analysis

To answer the question of whether the simulated price series exhibits persistence, we first examine the empirical power spectrum. We calculated the Welch-averaged spectral density of the simulated price series with 100,000 data points, with a Hanning window 100 units wide. The smoothed spectrum with 95% confidence bands is shown in Figure 8.

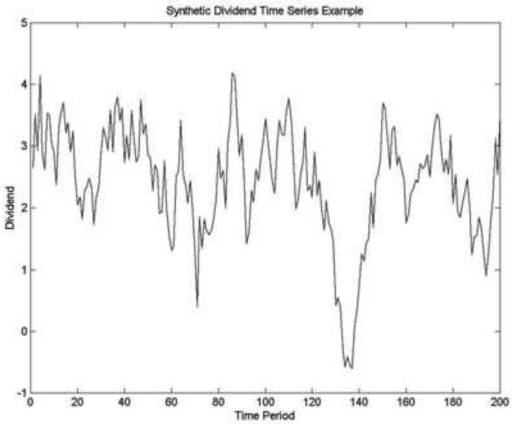

Fig. 5. Sample Dividend Series from Synthetic Data, 200 Periods

Fig. 6. Wage and Manager Effort Series from Synthetic Data

The spectrum indicates considerable power in the lower frequencies, a hallmark of long-memory processes. However, it does not appear that the spectrum demonstrates strong long-memory persistence. Rather, it appears at first blush that the spectrum of the second-best price series is still dominated by the AR(1) structure of the dividend process. In order to ascertain if long-memory has any impact we turn to a 'rough and ready' test of persistence given by [10]. This test attempts to verify whether or not the spectrum bears a some similarity to an inverse power law in frequency. In the literature on fractionally integrated time-series, this is often referred to as "1/f " noise.

To test the hypothesis that the observed data behave as 1/f noise, we regressed the log of the power spectrum against the log of the (sin of) fre-

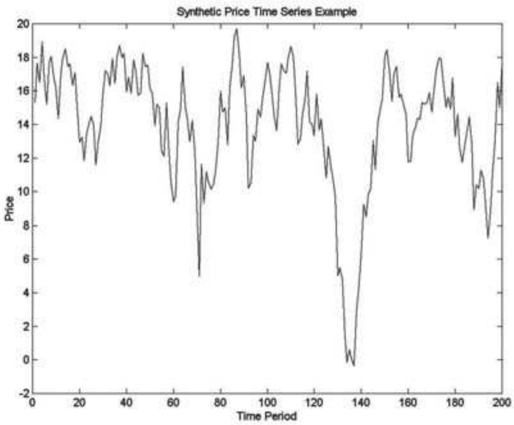

Fig. 7. Sample Price Series from Synthetic Data, 200 Periods

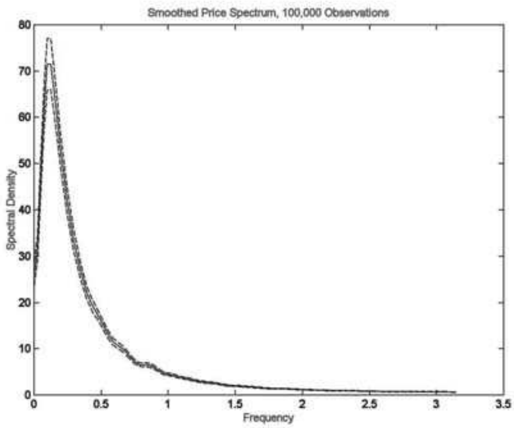

Fig. 8. Smoothed Price Spectrum with 95% Confidence Bands, 100,000 Obs.

quency, and the results are presented in Table 1. As shown in the table, the estimated long-memory exponent $d = 0.61$ is significant beyond the 99% confidence level. We conclude, then, that the simulated price series exhibits some long-memory persistence.

This results raises the obvious question as to what causes the observed persistence. One way of approaching this is to consider an experiment first performed in the 1940's by the hydrologist Harold E. Hurst. Hurst's experiment involved generating random sequences of biased random walks, and analyzing the properties of the resulting time-series. Subsequences were characterized by a fixed bias in the step size of the random walk. A second random variable determined when the bias would be changed, generating a new sub-

Table 1. Spectral Regression (Geweke & Porter-Hudak) Estimate Results

Variable	Coefficient	Std. Error	t-statistic
intercept	2.2096	0.0705	31.349**
d	0.6088	0.0441	13.816**
R^2	0.7578		

sequence. When Hurst examined large data sets generated in this fashion, he found significant evidence of long-memory persistence. Subsequent work on time-series models with exogenous structural breaks has confirmed the long-memory properties of the time-series generated by such models (see e.g. [24]). In our model, structural breaks in the trend of the stock price time series occur when the drift parameter swings from negative to positive (or vice-versa) in response to increased (or decreased) effort by the agent. These breaks occur endogenously (but randomly, given their dependence on the innovation process of the dividend) and, we believe, generate the observed persistence. As we will see in the following section, the trend breaks may also generate the significant conditional heteroskedasticity observed in the simulated price series.

5.2 Estimation of the GARCH (1,1) Model

The GARCH (1,1) model for conditional heteroskedasticity is defined as

$$p_t \mid \Psi_{t-1} \sim N\left(\beta \mathbf{p}_{t-l}, \sigma_t^2\right),$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \sigma_{t-1}^2,$$

$$\varepsilon_t = p_t - \beta \mathbf{p}_{t-l},$$

where Ψ_{t-1} is the information set available at time $t-1$, $\mathbf{p}_{t-l} \equiv [1, p_{t-1}, \ldots p_{t-l}]'$ is an $(l+1)-$dimensional vector of lagged endogenous variables, ε_t is the residual of the mean regression, and β, $\alpha \equiv [\alpha_0, \alpha_1, \alpha_2]$ are $(1 \times l + 1)$ and $(1 \times 3)-$dimensional vectors, respectively. This specification allows for the possibility of 'booms' and 'busts' (or alternatively, 'fads' and 'bubbles') in the price sequence, and has (along with its variants E-GARCH and N-ARCH) been implemented extensively on economic data (see e.g. [4], [15] and [23]).[3]

The GARCH (1,1) model of the second-best price series was estimated using Maximum Likelihood with 100,000 observations, with a first-order autoregressive process for the conditional price expectation. Before estimation,

[3] Veronesi (1996) also used the GARCH (1,1) specification to test for ARCH, in a model which uses 'regime shifts' (similar to the trend breaks of Perron [1989]) to generate correlated volatility in asset returns.

the Lagrange Multiplier (LM) test for ARCH ([7]) and the Jarque-Bera normality test were applied to the residuals of the mean equation. The results of the estimation and tests are presented in Table 2. The LM test strongly rejected the i.i.d. residual hypothesis at the 99% confidence level, while the Jarque-Bera test rejected the normality hypothesis beyond the 99% confidence level. These are strong indications that the time series contains some measure of correlated volatility.

Table 2. GARCH(1,1) Estimation Results

Variable	Coefficient	Std. Error	z-statistic
Mean Equation: $E_{t-1}p_t = \beta_0 + \beta_1 p_{t-1}$			
Constant	2.5293	0.0266	95.015^{**}
p_{t-1}	0.8309	0.0018	453.23^{**}
LM ARCH Test: $T * R^2 = 488.27$, $\Pr(i.i.d.) < 10e^{-6}$			
Jarque-Bera Statistic: 3815.3, $\Pr(normal) < 10e^{-6}$			
Variance Equation $E_{t-1}\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \sigma_{t-1}^2$			
Constant	0.4802	0.0243	19.739^{**}
ε_{t-1}^2	0.0631	0.0025	25.561^{**}
σ_{t-1}^2	0.7738	0.0097	79.328^{**}

** refers to significance at the 99% confidence level.

The coefficients of the GARCH (1,1) model were all statistically significant beyond the 99% confidence level. In addition, the conditional variance process is strongly persistent (with $\alpha_2 = 0.774$). This provides reasonable grounds for acceptance of the GARCH (1,1) model as demonstrating the existence of conditional heteroskedasticity in price data which is generated by second-best contracting. Naturally, fitting the GARCH model to the data is an *a priori* model misspecification, since the price data are actually generated by (14). The GARCH estimation is in this case simply used to demonstrate the strong presence of correlated volatility in the price data.

6 Conclusion

We have seen that the simple model presented here yields a price sequence which can have hidden nonlinear behavior. In addition, it appears that examples of this sequence are fit well by an ARCH-like specification, which has been noted to exist in empirical data. These results support the conclusion that endogenous correlated volatility and persistence is possible when the manager influences the production process and is in turn affected by his contractual relationship with the owner.

The model of owner-manager behavior in an asset-pricing model presented here has been simplified in many key ways. First and most important, the type of contract that the manager is able to make with the owner is essentially static–commitment only occurs over the single period the contract is in force, and the same contract is assumed to be accepted in perpetuity. Further investigation would see whether a more general model, incorporating a multi-period contract with commitment, would be substantively different from the simplified model presented here. It would be interesting to see, for example, whether the correlation between dividends and prices is less strong under a fully dynamic model, since it is empirically observed that dividend time series is less volatile than (hence, less strongly correlated with) the asset price series.

In addition, it is not clear whether the manager affects the mean of the production process (as assumed here), the variance of the process (as would be true, for instance, if the manager could influence the impact of the exogenous shock upon the dividend), or both. Future research will develop a production-based model of manager effort which can then suggest a dividend process such as equation (1) as a direct conclusion. Of course, given the analytical complexity of the simplified case, it is not at all clear that a more general model with the above extensions would yield analytically testable conclusions. Rather, it would appear from this presentation that the tools of numerical approximation and simulation of time series would be just as valuable for these cases.

7 Appendix

7.1 Solution of the First-Best Contract

Consider the first-best economy:

$$v^*(x_{t-1}) = \max_{w,a} \left\{ a + \rho x_{t-1} - \int w(x) dG(x|x_{t-1}, a) + \beta \int v^*(x) dG(x|x_{t-1}, a) \right\}$$

$$s.t. \int e^{-w(x)} dG(x|x_{t-1}, a) + e^a \leq |\overline{u}| ,$$

where we have used the functional forms from Definitions 1 and 2. From the optimization with respect to w we know that the wage payment will be independent of the observed output x, so that

$$e^{-w^*} + e^a = |\overline{u}| \Rightarrow$$

$$w^* = -\ln(|\overline{u}| - e^a),$$

given a particular level of managerial effort a.

The owner's optimization is now

$$v^*(x_{t-1}) = \max_a \left\{ a + \rho x_{t-1} + \ln(|\overline{u}| - e^a) + \beta \int v(x) dG(x|x_{t-1}, a) \right\}.$$

The objective function of the owner is concave in a. Posit a candidate policy function $a = a^*$, where a^* is some constant, and a candidate value function $v^*(x_{t-1}) = c_1 + c_2 x_{t-1}$. Then Bellman's equation at the optimal level of effort is

$$c_1 + c_2 x_{t-1} = a^* + \rho x_{t-1} + \ln(|\overline{u}| - e^{a^*}) + \beta c_1 + \beta c_2 (a^* + \rho x_{t-1}).$$

Matching coefficients yields

$$c_1 = a^* + \ln(|\overline{u}| - e^{a^*}) + \beta c_1 + \beta c_2 a^*$$

$$c_2 = \rho + \beta \rho c_2$$

or

$$c_1^* = \frac{1}{1-\beta} \left(\frac{a^*}{1-\beta\rho} + \ln(|\overline{u}| - e^{a^*}) \right),$$

$$c_2^* = \frac{\rho}{1-\beta\rho}$$

(note that this can also be directly verified by appealing to the equivalent sequence problem–cf. [28]).

The owner thus wishes to solve

$$\max_{a^* \in (-\infty, \ln(|\overline{u}|))} \left\{ a^* + \rho x_{t-1} + \ln(|\overline{u}| - e^{a^*}) + \frac{\beta}{1-\beta} \left(\frac{a^*}{1-\beta\rho} + \ln(|\overline{u}| - e^{a^*}) \right) + \frac{\beta\rho}{1-\beta\rho} a^* \right\}. \tag{17}$$

This problem has (after some rewriting) the associated first-order condition

$$\frac{e^{a^*}}{|\overline{u}| - e^{a^*}} = \frac{1}{1-\beta\rho} \Rightarrow$$

$$a^* = \ln\left(\frac{|\overline{u}|}{2-\beta\rho} \right).$$

This implies that the optimal wage payment is

$$w^* = -\ln(|\overline{u}| - e^{a^*}) = -\ln\left(|\overline{u}| \left(\frac{1-\beta\rho}{2-\beta\rho} \right) \right).$$

In the first-best equilibrium, then, the value of the firm is a linear function of the previous level of the dividend, while the wage and manager effort are constant. As either the rate of discounting β or the autoregressive parameter ρ

converge to zero the effort level converges to $\ln\left(\frac{|\overline{u}|}{2}\right)$, while the wage converges to $-\ln\left(\frac{|\overline{u}|}{2}\right)$. In these cases either the owner does not care about the future, or the manager does not contribute to future dividends through the past dividend level, and the manager's effort level is low. As β or ρ rise, however, the effort level rises and the wage rises to compensate.

7.2 Existence of the Second-Best Value Function

We seek to prove that a function v^* exists which solves:

$$v(x) = \max_{w,a^*} \left\{ \int_X (y - w(y))g(y\,|x, a^*)dy + \beta \int_X v(y)g(y\,|x, a^*)dy \right\}$$

such that

$$a^* \in \arg\max_a \int_X u(w(y))g(y\,|\,x, a)dy - h(a),$$

$$\int_X u(w(y))g(y\,|x, a^*)dy - h(a^*) \geq \bar{u},$$

where X is the dividend space, y is the current (unobserved) dividend, x is the previous dividend, w is the wage paid to the manager, a^* is the manager's optimal action, and $\bar{u}$ is the manager's reservation utility. Note that for exposition our notation here differs slightly from the form in the text–in addition, we have replaced the current dividend density function dG with the density function gdy.

We assume that the first-order approach is valid; that is, the manager's preferences are such that the incentive compatibility condition (ICC)

$$a^* \in \arg\max_a \int_X u(w(y))g(y\,|\,x, a)dy - h(a)$$

may be replaced with (cf. [18])

$$\int_X u(w(y))\frac{\partial g(y\,|\,x, a^*)}{\partial a}dy - h'(a^*) = 0. \tag{ICC}$$

Assuming that the first-order approach is valid implies that an interior solution to the problem ICC exists, i.e. that the second-order condition satisfies

$$\int_X U(w(y))\frac{\partial^2 g(y\,|\,x, a^*)}{\partial a^2}dy - h^{''}(a^*) < 0.$$

when the second-order condition exists. In order to ensure this, we must add the condition that the density function of current dividends $g\,(y\,|\,x, a)$ be at least twice-continuously differentiable in a.

From this we know immediately that the Implicit Function Theorem (IFT) applies around a^*; next we assume that $g\,(y\,|\,x, a)$ is at least twice-continuously differentiable in x so that we may write

$$\int_X U(w(y)) \frac{\partial g(y \mid x, a^*(x;w))}{\partial a} dy - h'(a^*(x;w)) = 0.$$

Note that since w takes as its argument the current dividend y, the optimal action a^* will not depend parametrically upon w, but rather functionally. By the IFT, however, we know that this functional dependence is one-to-one. Thus, the incentive compatibility condition defines the optimal action given the previous dividend and the wage function.

We may now write the dynamic programming problem of the owner as

$$v(x) = \max_w \left\{ \int_X (y - w(y)) g(y \,| x, a^*(x;w)) dy + \beta \int_X v(y) g(y \,| x, a^*(x;w)) dy \right\} \tag{18}$$

such that

$$\int_X u(w(y)) g(y \,| x, a^*(x;w)) dy - h(a^*(x;w)) \geq \overline{u} \tag{PC}$$

where PC is the participation constraint of the manager.

This problem has a straightforward solution. The current-period return function is bounded in w by assumption, and we also suppose that the conditional expected value of the current dividend is finite (i.e., $\int_X yg(y \,| x, a^*(x;w) dy < \infty$). The constraint PC is compact-valued, non-empty and continuous. As before (see equation 12) we define an operator $T : C_X^1 \to C_X^1$ by

$$T(v)(x) = \int_X (y - w^v(y)) g(y \,| x, a^*(x;w^v)) dy + \beta \int_X v(y) g(y \,| x, a^*(x;w^v)) dy \tag{19}$$

where w^v is the optimal solution to the problem (18) + (PC) for a given function v.

From the above considerations we know that the Theorem of the Maximum obtains–the T operator takes bounded continuous functions into bounded continuous functions. Furthermore, we can use Blackwell's sufficiency conditions to show that T is a contraction. Recall that if T is an operator taking bounded continuous functions into bounded continuous functions, then it is a contraction mapping if

1. T is monotonic, i.e. $T(v)(x) \leq T(w)(x)$ whenever $v(x) \leq w(x)\ \forall x$
2. T is 'sublinear', i.e. $T(v + c)(x) \leq T(v)(x) + \beta c\ \forall v, \beta \in (0,1), c < 0$.

It is clear that the operator defined by (19) satisfies Blackwell's conditions, so that T is a contraction mapping. Hence, we know that a continuous function v^* exists which solves the owner's problem, and that

$$\lim_{n \to \infty} T^n(v_0) = v^*\ \forall v_0 \in C_X^1.$$

7.3 Second-Best Approximation Details

The second-best value function v^* was estimated using a single-layer feedforward neural network of 6 hidden units. Following the procedure outlined in Section 3, neural networks were fit to dividend grid data and then used to calculate the next iteration of the value function. Each neural network fit the grid data so that the sum-of-squared-error (SSE) between the estimated data and the actual data was less than 10^{-10}.

The dividend space X ranged from -2 to 20, and was divided into a grid $\{x_i\}$ of 30 equidistant points to be used as network input. These points were used to calculate the value function points $v^n(x_i)$. These points constituted the target vector for fitting the neural network. The actual grid data used to estimate the neural network varied from iteration to iteration. An adaptive grid was used to focus the neural network's attention on those points which were particularly hard to estimate, i.e. those grid points whose absolute errors were above the mean absolute error of the current iteration's estimate. Convergence of the value function was assumed when the largest absolute error between consecutive value function estimates was less than 10^{-3}. As mentioned in the text, the owner's discount factor β was set to 0.9, the autoregressive parameter ρ of the dividend process was also 0.9, and the reservation utility $\bar{u}$ was -2.

After the value function iteration had converged, grid data for the optimal wage function and the optimal effort function were generated. The optimal wage function was estimated using a single-layer feedforward neural network of 16 hidden units, and the SEE between the estimated and actual wage values was less than 10^{-4}. The optimal effort function required 6 hidden units, and the SSE between the estimated and actual effort values was less than 10^{-12}.

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Central-Bank Interest-Rate Control in a Cashless, Arrow-Debreu Economy

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Summary. A pure-exchange, competitive economy with date-specific units of account is studied. It differs from the standard pure-exchange model in having unit-of-account endowments. For such an economy, conditions are given for an outside agent, a central bank, to be able to control the nominal rate of interest. It can do so provided it has positive unit-of-account endowments. A more general outside agent that has a positive endowment of some good and positive unit-of-account endowments can select the time path of the price level.

1 Introduction

Money is often described as having three functions: (i) a unit-of-account function, (ii) a medium-of-exchange function, and (iii) a store-of-value function. In a cashless economy, the third is not operative and, probably, neither is the second. To lay people, the notion of a cashless economy may seem strange. To economists, it is not at all strange. After all, the theory of relative prices, the modern competitive version of which is called the Arrow-Debreu model, has always been the developed part of economics. It is monetary theory that has always been undeveloped. And, as it happens, the Arrow-Debreu model has room for a unit of account—what is called an abstract unit of account, abstract in the sense of not having a physical counterpart. This note describes conditions under which an activity that resembles central-bank open-market operations determines nominal interest rates in a simple Arrow-Debreu model.

2 A two-date model

The model is of a 2-date, pure-exchange economy with N people and L goods at each date. In order to make the model fit the unit-of-account vision that people seem to have in mind for a cashless economy, my version has date-specific abstract units of account called date-1 ecus and date-2 ecus. Goods are indexed by $l \in \{1, 2, ..., L\}$ and people by $n \in \{1, 2, ..., N\}$. Let $\omega_t^n \in \mathbb{R}_{++}^L$ denote person n's endowment vector of date-t goods. Although somewhat non standard, I will also endow people with ecus at each date and let $e_t^n \in \mathbb{R}$ denote person $n's$ endowment of date t ecus. (I permit e_t^n to be positive or negative. A negative holding is a debt denominated in date-t ecus.) Note that all the objects are perishable. For goods, that is what pure exchange means. For ecus that is one of the meanings of cashless: there is no *technology* that permits a person to convert ecus at one date into ecus at another date.

Let $c_t^n \in \mathbb{R}_+^L$ be n's consumption vector of date t goods. Person n has a utility function, a function of consumption of goods, $u^n : \mathbb{R}_+^{2L} \to \mathbb{R}$, which is strictly increasing. Also, let x_t^n denote person $n's$ consumption or end-of-date-t holdings of date-t ecus. As implied by the function u, people do not value consumption of ecus, just as they do not value consumption of cash in a cash economy.

Definition 1. *Person n can afford non negative $(c_1^n, c_2^n, x_1^n, x_2^n)$ at the prices (p_1, p_2, R) if*

$$(p_1\omega_1^n + e_1^n - p_1c_1^n - x_1^n) + R(p_2\omega_2^n + e_2^n - p_2c_2^n - x_2^n) \geq 0. \qquad (1)$$

The restriction $x_t^n \geq 0$ should be interpreted as a no-reneging condition. If $x_t^n < 0$, then a person leaves date t owing date-t ecus. That is not permitted. Notice that the nominal interest rate is $\frac{1}{R} - 1$. Given that definition of what a person can afford, we can define a competitive equilibrium (CE) as follows.

Definition 2. *A CE is $(c_1^n, c_2^n, x_1^n, x_2^n)$ for each n and (p_1, p_2, R) such that (i) $(c_1^n, c_2^n, x_1^n, x_2^n)$ maximizes u^n subject to being affordable at (p_1, p_2, R), and (ii) the allocation is feasible; that is, $\sum_n c_t^n \leq \sum_n \omega_t^n$ and $\sum_n x_t^n \leq \sum_n e_t^n$ for $t = 1, 2$.*

This definition does not include equilibria in which either or both of date-1 ecus and date-2 ecus are worthless. In general, there are such equilibria. Therefore, when we describe necessary conditions for a CE that satisfies definition 2, they are necessary conditions for a CE in which date-1 and date-2 ecus have value. Consideration of such equilibria is similar to consideration of valued-cash equilibria in models with cash. Often, those models also have equilibria in which cash is not valued.

If there are no endowments of the unit of account—that is, if $e_t^n \equiv 0$—then this is (a special case of) the standard pure-exchange model. And it has the standard zero-degree homogeneity property.

Claim. If (c_1^n, c_2^n) for each n and (p_1, p_2, R) is a CE, then (c_1^n, c_2^n) for each n and $(\gamma_1 p_1, \gamma_2 p_2, \gamma_1 R/\gamma_2)$ is a CE for any $(\gamma_1, \gamma_2) \in \mathbb{R}_{++}^2$.

Proof. Obvious.

Notice that this implies that there is a sense in which $R \in \mathbb{R}_+$ can be selected arbitrarily. Such a selection can be interpreted as influencing the inflation rate between dates 1 and 2 because R has the units date-1 ecus per unit of date-2 ecus.

Now suppose that there are endowments of ecus. Then, for some prices, some people may have negative wealth. That may cause problems for existence of a CE. If a CE exists for given ecu endowments, and if those endowments are held fixed, then there would seem to be what has been called real indeterminacy because as prices vary the wealth distribution varies. I will duck the existence question here and deal with properties of a CE.

Claim. In any definition 2 CE, (i) prices are positive ($p_1 > 0, p_2 > 0$, and $R > 0$) and $x_t^n \equiv 0$ and (ii) the feasibility conditions hold at equality and $\sum_n e_t^n = 0$.

Proof. Part (i) follows from monotonicity of u. As for part (ii), monotonicity of u also implies that (1) holds at equality, and, therefore, that the sum of (1) over n holds at equality. Then, because prices are positive, the sum of (1)) over n implies that the feasibility conditions hold at equality. That and $x_t^n \equiv 0$ imply $\sum_n e_t^n = 0$.

The necessary condition, $\sum_n e_t^n = 0$, is reassuring. It says that total endowments of ecus at each date are zero. In other words, claims on date-t ecus must be offset by debts of date-t ecus.

And, as is standard, if we adjust ecu endowments appropriately, then the zero-degree homogeneity property holds.

Claim. If (c_1^n, c_2^n) for each n and (p_1, p_2, R) is a CE for ecu endowments e_t^n, then (c_1^n, c_2^n) for each n and $(\gamma_1 p_1, \gamma_2 p_2, \gamma_1 R/\gamma_2)$ is a CE for ecu endowments $\gamma_1 e_1^n$ and $\gamma_2 e_2^n$ for any $(\gamma_1, \gamma_2) \in \mathbb{R}_{++}^2$.

Proof. Obvious.

3 A Central Bank

Subject to the above proviso about existence, it seems that the nominal interest rate can be anything. Therefore, there would seem to be scope for an outside entity, the central bank, to choose the nominal interest rate. One way to think of a central bank as enforcing a particular magnitude for the nominal interest rate is to have it offer to buy date-1 ecus for date-2 ecus, and vice versa. This would seem to be the analogue of open-market operations.

To explore this, let us label the central bank agent 0. Assuming, as is natural, that the central bank does not deal in goods and does not have endowments of goods, we can regard it as choosing non negative x_1^0 and x_2^0 subject to the following special case of (1):

$$e_1^0 - x_1^0 + R(e_2^0 - x_2^0) = 0. \tag{2}$$

We can interpret $e_t^0 - x_t^0$ as the central bank's sale (purchase if negative) of date-t ecus. Then, (2) says only that the central bank trades at a price.

Because the central bank does not maximize utility in any ordinary sense, we do not assume that it necessarily chooses $x_t^0 = 0$. However, we do assume that it is also subject to $x_t^0 \geq 0$: it cannot leave date t owing date-t ecus.

Because the central bank does not deal in goods and does not maximize utility, the only amendment needed in the definition of a CE is to replace the feasibility condition for ecus to include agent 0. And because (1) holds at equality for each private agent and (2) holds, it follows that such feasibility implies $\sum_{n=0}^{N} x_t^n = \sum_{n=0}^{N} e_t^n$ for $t = 1, 2$. And because $x_t^n = 0$ for $n \neq 0$ in any CE, a necessary condition for a CE is

$$x_t^0 - e_t^0 = \sum_{n=1}^{N} e_t^n \text{ for } t = 1, 2. \tag{3}$$

Notice that for general private endowments of ecus, endowments that do not satisfy $\sum_{n=1}^{N} e_t^n = 0$, (2) and (3) are inconsistent. Therefore, we continue to assume that $\sum_{n=1}^{N} e_t^n = 0$, the necessary condition for a CE when there is no central bank. Under that assumption and the assumption that the central bank has positive endowments of ecus, an assumption discussed below, it can be shown that the central bank can make an arbitrarily chosen R a necessary condition for a CE.

Claim. Assume $\sum_{n=1}^{N} e_t^n = 0$ and $e_t^0 > 0$ for $t = 1, 2$. For any $\hat{R} \in \mathbb{R}_{++}$, the central bank (agent 0) can behave as a price taker in such a way that a necessary condition for a CE is $R = \hat{R}$.

Proof. We have to construct behavior for agent 0. Let the function $f(R)$ determine the choice of $e_1^0 - x_1^0$, with $e_2^0 - x_2^0$ given by (2). Let $f : \mathbb{R}_+ \to (-\hat{R}e_2^0, e_1^0)$ be strictly monotone, continuous, and satisfy $f(\hat{R}) = 0$. The bounds on the range of f are chosen to satisfy $x_t^0 \geq 0$. Now, suppose, by way of contradiction, that there is a CE with $R \neq \hat{R}$. By the choice of f, this implies $x_t^0 - e_t^0 \neq 0$. But this violates (3), a necessary condition for a CE.

There seems to be nothing more to this proof than the idea that if the central bank chooses to supply something for which private excess demand is perfectly inelastic at the quantity 0, then equilibrium requires that the price be such as to make the central bank willing not to supply it.

Positive endowments for the central bank play an important role. In particular, if the central bank has no endowments, then its budget constraint, (2), and $x_t^0 \geq 0$ imply $x_t^0 - e_t^0 = 0$ at all R. How do we interpret positive endowments for the central bank simultaneously with $\sum_{n=1}^N e_t^n = 0$? I am not sure. Perhaps the positive central-bank endowments are the analogue of the central bank's power in a cash economy to print cash at any time.

It may seem odd to claim that the central bank *can* determine the nominal interest rate without providing a proof that there exists such an equilibrium. The known existence results are for versions without endowments of the date-specific units of account, versions in which the magnitude of the nominal interest rate does not matter in the above model. Almost certainly, those existence results can be extended. After all, by choosing p_1 and p_2 to be sufficiently large, given unit-of-account endowments can be made arbitrarily small in real terms. And setting a nominal interest rate does not prevent p_1 and p_2 from being arbitrarily large.

4 Fiscal and monetary policy

Notice, of course, that control of R as just described leaves one degree of indeterminacy. It is tempting, therefore, to describe a somewhat more general policy, one that would eliminate all indeterminacy. Now, I assume that agent 0 has, in addition to endowments of ecus, some of one of the goods. Without loss of generality, let it have an endowment of good 1 at date 1, denoted ω_{11}^0. I will not describe where it obtained this endowment. One can think of it as coming from a bit of lump-sum taxation, which is why I call the policy being described fiscal *and* monetary policy. Just as the agent 0 ecu endowments only had to positive, not large, I only require that ω_{11}^0 is positive.

Now I can show that there is a way for agent 0 to behave, behave in the sense of having an excess demand function, that allows it to determine both p_{11}, the price of good $one - one$, and R. Before doing that, I have to again amend the definition of a CE. Once again, agent 0 does not maximize anything. Hence, we need only amend the feasibility conditions to include agent 0. I assume that agent 0 consumes at most some of good $one - one$. As a convention, I define its consumption, denoted c_{11}^0, to be the amount of good $one - one$ it has after trading. Then the feasibility conditions hold at equality.

Claim. Assume $\sum_{n=1}^N e_t^n = 0$ and $e_t^0 > 0$ for $t = 1, 2$ and $\omega_{11}^0 > 0$. For any $(p_{11}^*, R^*) \in \mathbb{R}_{++}^2$, agent 0 can behave as a price taker in such a way that a necessary condition for a CE is $(p_{11}, R) = (p_{11}^*, R^*)$.

Proof. We construct behavior for agent 0. Let the function $f(p, R) : \mathbb{R}_+^{2L+1} \rightarrow \mathbb{R}^{2L+2}$ denote the excess demand for agent 0. We suppose that $f(p, R) = f(p', R')$ if $p_{11} = p_{11}'$ and $R = R'$. Moreover, $f(p, R) \equiv 0$ except for good $one - one$ and ecus. Aside from continuity, we let the sign of excess demand for the vector (f_{11}, f_2), excess demand for good $one - one$ and for date-2 ecus be as indicated in figure 1 with f_1, excess demand for date-1 ecus, determined by the

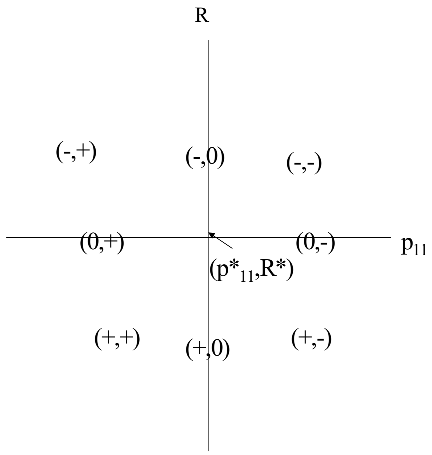

Fig. 1. Sign of (f_{11}, f_2)

agent-0 budget constraint at equality. In addition, let f be continuous. Now, suppose, by way of contradiction, that there is a CE with $(p_{11}, R) \neq (p_{11}^*, R^*)$). By the choice of f, this implies $x_t^0 - e_t^0 \neq 0$ for some t. But this violates (3), a necessary condition for a CE.

The idea is the same as in the proof of claim 4. The private sector cannot be either a net supplier or demander of ecus at either date. Therefore, given that agent 0 is a net supplier or demander at prices that do not satisfy $(p_{11}, R) = (p_{11}^*, R^*)$, satisfaction of this equality is necessary for a CE. By the way, while the behavior posited in the proof of claim 4 is consistent with a CE in which ecus at both dates are worthless, that is not the case for the behavior posited in the proof of claim 5. As should come as no surprise, the willingness of agent 0 to sell goods for ecus prevents ecus from being worthless.

Finally, it is obvious that nothing in these arguments hinges on there being only two dates. The results apply for any finite number of dates.

5 Concluding remarks

The above model is disarmingly simple. It can be because it does not have to confront the hard problems of modeling cash economies. Because no one holds cash from one date to the next, there is no need to explain why people hold non interest-bearing cash when they seem to have available assets with higher rates of return. Also, there is no zero lower bound on nominal interest rates. This bound arises in an economy with cash because people have available a technology for converting cash at the current date into cash in the future; namely, storing it under their mattresses. In a cashless economy, there is no such opportunity and, therefore, no lower bound on nominal interest rates. Also, for cashless economies, there is no terminal condition problem concerning why cash has value at a last date.

But simplicity it not an end in itself. Suppose we were to require not only that the model be cashless, but that it be a limit of a cash economy, a limit as something in the economy makes cash disappear. If we take that sensible view, then we have to face all the hard problems of monetary theory: we have to decide what cash (monetary) model we like and what limit to take to produce cashlessness.

In that regard, some recent work on monetary models takes a mechanism-design point of view and requires that money play a benefical role relative to all feasible mechanisms. That work concludes that such a monetary model must have several frictions relative to Arrow-Debreu. For example, in [1], individuals cannot commit to future actions and there is imperfect public monitoring of past individual actions in the form of an updating lag of the public record of individual actions. Remove either and you get a cashless economy. However, if you only remove the imperfect monitoring by letting the lag get short, which is what seems to be happening in actual economies, then you get a cashless economy that is not an Arrow-Debreu model. Thus, it should not be taken for granted that the cashless limit of interest is an Arrow-Debreu model.

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